GEOMETRY OF LEAF SPACES OF SINGULAR FOLIATIONS

by

Ian David Seiji Miyamoto

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ABSTRACT

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This thesis has three parts. First, we survey Stefan and Sussmann's work on singular foliations, highlighting diffeological objects that arise. We then propose a transverse equivalence of singular foliations, and show: equivalent foliations have diffeomorphic leaf spaces; the converse need not hold; and regular foliations with Hausdorff holonomy groupoid are transverse equivalent if and only if they are Morita equivalent.

Second, we show that for a singular foliation $(M, \mathcal{F})$, the quotient $\pi : M \to M/\mathcal{F}$ induces an isomorphism $\pi^* : \Omega^\bullet(M/\mathcal{F}) \to \Omega_b^\bullet(M, \mathcal{F})$, where $\Omega_b^\bullet(M, \mathcal{F})$ denotes the complex of $\mathcal{F}$-basic forms on M, when: $\mathcal{F}$ is regular; when the union of k-leaves is a diffeological submanifold, for all k; and when π^* is an isomorphism if we excise the 0-leaves.

Finally, we introduce diffeological quasifolds and quasifold groupoids. A quasifold is locally modelled by the affine action of countable groups Γ on $\mathbb{R}^n$. When the Γ are finite, we recover orbifolds. We show that the categories of diffeological quasifolds and quasifold groupoids are equivalent, when we restrict the arrows to local diffeomorphisms and locally invertible bibundles, respectively.

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CONTENTS

PUBLICATIONS

KM22 Yael Karshon and David Miyamoto, *Quasifold groupoids and diffeological quasifolds*, Submitted preprint. https://arxiv.org/abs/2206.14776. 2022.

Miy23a David Miyamoto, *The basic de rham complex of a singular foliation*, Int. Math. Res. Not. IMRN (2023), no. 8, 6364-6401. MR 4574377

Miy23b David Miyamoto, *Singular foliations through diffeology*, Accepted preprint. https://arxiv.org/abs/2303.07494. To appear in Contemporary Mathematics (proceedings) for the special session "Recent advances on diffeologies and their applications" held at the AMS-SMF-EMS Joint International Meeting in Grenoble, July 2022, published by the American Mathematical Society. 2023.

1 INTRODUCTION

On the nLab page for "generalised smooth spaces," we find the charming aphorism:

> Manifolds are fantastic spaces. It's a pity that there aren't more of them. [1]

Indeed, it is quite easy to inadvertently exit the category of smooth manifolds. Consider, for instance, the torus $T^2 := \mathbb{R}^2/\mathbb{Z}^2$, viewed as the quotient of $\mathbb{R}^2$ by the integer lattice $\mathbb{Z}^2$. Let $\tilde{L}_\theta$ be a line in $\mathbb{R}^2$ through the origin, with slope θ. Then $L_\theta := \tilde{L}_\theta/\mathbb{Z}^2$ is a Lie subgroup of T^2, so we may consider the quotient space $T_\theta := T^2/L_\theta$. If the slope θ is rational, then L_θ is a circle, and T_θ is also a circle. But if θ is irrational, then L_θ is isomorphic to the Lie group $\mathbb{R}$, and T_θ - which we call an *irrational torus* - is far from a manifold: its topology is trivial.

To assuage this seemingly abrupt dependence on θ, we place the T_θ into the category of diffeological spaces. Formally introduced by Souriau in 1980, but closely related to structures considered by Chen in the 1970s, a *diffeological space* is a set X equipped with a collection of maps (called *plots*) from open subsets of Cartesian spaces into X satisfying three axioms (Definition 2.1). These plots take the role of charts of smooth manifolds, thus giving a notion of smooth maps between diffeological spaces, and we can form the category of diffeological spaces and smooth maps between them. Subsets, quotients, and function spaces of diffeological spaces all carry natural diffeologies, and manifolds are naturally diffeological spaces. In particular, the T_θ all inherit the quotient diffeology from T^2 regardless of the slope θ.

The benefit of viewing T_θ as diffeological spaces comes from the following result, proved by Iglesias-Zemmour and Donato in 1985 [DI85].

Theorem. *The diffeology of T_θ is not trivial. Moreover, T_θ and $T_{\theta'}$ are diffeologically diffeomorphic if and only if*

$$\theta' = \frac{a\theta + b}{c\theta + d} \tag{1.1}$$

1 https://ncatlab.org/nlab/show/generalized+smooth+space

for some $a, b, c, d \in \mathbb{Z}$ *with* $ad - bc = \pm 1$.

An alternative approach to the irrational tori is through "higher structures," by which we mean Lie groupoids or (singular) foliations:

- we may encode the data of the action of L_θ on T^2 in the action groupoid $G_\theta := L_\theta \ltimes T^2 \rightrightarrows T^2$;
- or we may consider the regular foliation $\mathcal{F}_\theta$ given by the partition of T^2 into the orbits of L_θ.

More generally, every Lie groupoid $G \rightrightarrows M$ or foliation $(M, \mathcal{F})$ sits above a natural diffeological quotient space. For a Lie groupoid, we consider its space of orbits M/G, and for a foliation, we consider its leaf space $M/\mathcal{F}$. The motivating question of this thesis is:

Question. What information about a higher structure on a manifold M (a Lie groupoid $G \rightrightarrows M$ or a singular foliation $(M, \mathcal{F})$) can we recover from the underlying diffeological quotient space?

We approach this question in three ways: through notions of transverse equivalence of singular foliations, through the basic complex of a singular foliation, and through the notion of diffeological quasifolds and quasifold groupoids. We describe these approaches in the chapter overview below.

1.1 CHAPTER OVERVIEW

The contents of this thesis are derived from a paper and two submitted preprints, namely, in order of completion during my degree: [Miy23a], [KM22], and [Miy23c]. These works are self-contained, and therefore admit some overlap between them. In order to avoid the most egregious repetition, in Chapter 2 we review the elements from the study of diffeology, Lie groupoids, and regular foliations that are common to all three works. Sections 2.1 and 2.2 are drawn from [KM22], and Section 2.3 from [Miy23a]. We emphasize that the notion of a locally invertible bibundle (Definition 2.30) is new, and that Section 2.3 covers properties of the holonomy groupoid that are not often spelled out in the literature.

Chapter 3 is essentially [Miy23c]. Here we introduce singular foliations, which are partitions of a manifold M into connected submanifolds of perhaps varying dimension, fitting together smoothly. We pay special attention to Stefan [Ste74] and Sussmann's [Sus73] contributions and their interaction with diffeology. Of particular note is our Proposition 3.26, which states that every singular foliation is induced by the orbits of a connected diffeological group acting on M. We then propose, in Definition 3.41, a notion of transverse equivalence of singular foliations, which we call *Molino transverse equivalence* after P. Molino, who considered this equivalence in the regular case. In Proposition 3.43, we show that a Molino transverse equivalence of singular foliations descends to a diffeological diffeomorphism between their leaf spaces. In Corollary 3.50, we conclude that regular foliations are Molino transverse equivalent if and only if their holonomy groupoids are Morita equivalent, provided these holonomy groupoids are Hausdorff. In the case of irrational tori, the holonomy groupoid of $(T^2, \mathcal{F}_\theta)$ is the action groupoid G_θ, which is Hausdorff. Thus, by Corollary 3.50, the foliations $\mathcal{F}_\theta$ and $\mathcal{F}_{\theta'}$ are Molino transverse equivalent if and only if the action groupoids $L_\theta \ltimes T^2$ and $L_{\theta'} \ltimes T^2$ are Morita equivalent. By Proposition 3.43 (but also Proposition 2.32), this implies that T_θ and $T_{\theta'}$ are diffeomorphic, and in particular (1.1) must hold. We address the converse in Chapter 5. Our Molino transverse equivalence derives from Molino's ideas in [Mol88, Chapter 2], and is related to Garmendia and Zambon's Hausdorff Morita equivalence from [GZ19].

Chapter 4 gives the main result of [Miy23a]. It treats the problem of relating the complex of diffeological differential forms on the leaf space $M/\mathcal{F}$ to the complex of basic differential forms on $(M, \mathcal{F})$. A differential form α on M is *$\mathcal{F}$-basic* if, for every vector field X tangent to the leaves, both

$$\iota_X \alpha = 0 \text{ and } \mathcal{L}_X \alpha = 0.$$

The $\mathcal{F}$-basic forms assemble into a de Rham subcomplex $\Omega^\bullet_b(M, \mathcal{F})$ of $\Omega^\bullet(M)$. On the other hand, the diffeological leaf space $M/\mathcal{F}$ carries diffeological differential forms. The quotient map $\pi : M \to M/\mathcal{F}$ induces a pullback map $\pi^* : \Omega^\bullet(M/\mathcal{F}) \to \Omega^\bullet_b(M, \mathcal{F})$. Theorems 4.11, 4.20, and 4.27 describe three situations in which pullback π^* by the quotient map $\pi : M \to M/\mathcal{F}$ induces isomorphisms of these complexes, respectively:

- when $\mathcal{F}$ is a regular foliation;
- when the union of k-leaves of $\mathcal{F}$ is a diffeological submanifold of M, for all k;
- when π^* is an isomorphism if we excise the 0-leaves.

For the irrational tori, Theorem 4.11 shows that we have an isomorphism $\pi^* : \Omega^\bullet(T_\theta) \to \Omega^\bullet_b(T^2, \mathcal{F}_\theta)$. Some time after proving Theorem 4.11, we found that Hector, Macías-Virgós, and Sanmartín-Carbón came to the same conclusion in [HMVSC11]. We were not aware of their work at the time, and our techniques admit Corollary 4.12, a generalization of Theorem 4.11. Theorem 4.20 generalizes the results of Karshon and Watts [KW16] and Watts [Wat22]. Whether π^* is always an isomorphism remains an open question. We indicate directions for future research in Section 4.3.

Finally, Chapter 5 presents [KM22]. We introduce (the bicategories of) diffeological quasifolds and effective quasifold groupoids. These generalize orbifolds and orbifold groupoids; whereas orbifolds are locally modelled by linear actions of finite groups on $\mathbb{R}^n$, quasifolds are locally modelled by affine actions of countable groups on $\mathbb{R}^n$. Our Theorem 5.32 states that these two categories are isomorphic when we take the arrows to be local diffeomorphisms and locally invertible bibundles, respectively. In particular, the irrational tori T_θ are diffeological quasifolds, and the action groupoids G_θ are Morita equivalent to quasifold groupoids. Therefore Theorem 5.32 implies that the T_θ are diffeomorphic if and only if the G_θ are Morita equivalent, thus if and only if (1.1) holds. We end with an interesting non-example, and in Section 5.5 describe some research in progress. The preprint [KM22] is joint work with Yael Karshon, and the project was initiated in Masrour Zoghi's thesis [Zog10]. Iglesias-Zemmour and Prato also introduced diffeological quasifolds in [IZP21].

Chapters 3, 4, and 5 all rely on material in Chapter 2, but are otherwise independent. To facilitate their independence, the introductions to each chapter are those from the corresponding papers. These introductions are more comprehensive than this overview, and contain the necessary historical context and relevant literature.

2 DIFFEOLOGY, LIE GROUPOIDS, AND REGULAR FOLIATIONS

The first two sections of this review chapter are from [KM22], and the last is from [Miy23a].

2.1 THE BASICS OF DIFFEOLOGY

In this section, we provide a review of the basic concepts from diffeology that we will use throughout this thesis. Diffeology was introduced in the 1980s by J. M. Souriau, and K. T. Chen worked with similar structures in the 1970s. Our main reference is Iglesias-Zemmour's book [IZ13]. Note that this book has been updated and reprinted in [IZ22].

Definition 2.1 (Diffeology)**.** Let X be a set. A *parametrization* into X is a map from an open subset of a Cartesian space into X. A *diffeology* on X is a set $\mathcal{D}$ of parametrizations, whose members are called *plots*, such that

- constant maps are plots;
- if a parametrization $p : \mathsf{U} \to X$ is such that about each $r \in \mathsf{U}$, there is an open $\mathsf{V} \subseteq \mathsf{U}$ and a plot $q : \mathsf{V} \to X$ such that $p|_{\mathsf{V}} = q$, then p is a plot;
- if $p : \mathsf{U} \to X$ is a plot and V is an open subset of a Cartesian space, then for any smooth $F : \mathsf{V} \to \mathsf{U}$, the pre-composition F^*p is a plot.

A set equipped with a diffeology is a *diffeological space*.

The set of locally constant parametrizations into X, and the set of all parametrizations into X, are both diffeologies, called respectively *discrete* and *coarse*. Every other diffeology sits between these two. A classical smooth manifold[1] M carries a canonical diffeology $\mathcal{D}_M$ consisting of the smooth maps (in the usual sense) from subsets of Cartesian spaces into M.

1 A smooth manifold is a topological space equipped with a maximal smooth atlas that, unless we say otherwise, is Hausdorff and second-countable. If we use the term "manifold" unqualified, we mean a smooth manifold.

Definition 2.2 (Smooth maps)**.** We say a map $f : X \to Y$ between diffeological spaces is *(diffeologically) smooth* if for every plot p of X, the pullback $p^* f$ is a plot of Y. Denote the set of smooth maps from X to Y by $C^\infty(X, Y)$.

If X is discrete, or Y is coarse, all maps $X \to Y$ are smooth. A map between classical manifolds is diffeologically smooth if and only if it is smooth in the classical sense. Diffeological spaces and smooth maps between them form a category.

Definition 2.3 (Category of diffeological spaces)**.** The category **Diffeol** has objects diffeological spaces, and arrows smooth maps between them. When we need to view **Diffeol** as a bicategory, we simply add the identity 2-arrows.

Assigning to each classical smooth manifold M the diffeological space $(M, \mathcal{D}_M)$ defines a full and faithful functor from the category of classical manifolds, with their smooth maps, into **Diffeol**.

We will require a notion of locality for diffeological spaces.

Definition 2.4 (D-topology)**.** The *D-topology* on a diffeological space X is the finest topology in which all plots are continuous. Equivalently, $U \subseteq X$ is D-open if and only if $p^{-1}(U)$ is open for all plots p.

Every smooth map is continuous in the D-topology. The D-topology of a classical manifold M, viewed as a diffeological space, is its manifold topology.

Every subset of a diffeological space inherits a subset diffeology:

Definition 2.5 (Subset diffeology)**.** For a subset S of a diffeological space X, with inclusion denoted $\iota : S \hookrightarrow X$, the *subset diffeology* on S consists of all parametrizations $p : \mathsf{U} \to S$ such that $\iota \circ p$ is a plot of X.

Given a diffeological space X and a subset S, the D-topology of the subset diffeology on S is contained in the subset topology that S inherits from the D-topology on X. When S is a D-open subset of X, these topologies coincide. We use the D-topology and subset diffeology to define local diffeomorphisms:

Definition 2.6 (Local diffeomorphisms)**.** A map $f : X \to Y$ between diffeological spaces is a *local diffeomorphism* if, for each $x \in X$, the map restricts to a diffeomorphism between some D-open neighbourhoods of x and $f(x)$. Denote the set of local diffeomorphisms by $\mathrm{Diff}_{\mathrm{loc}}(X, Y)$.

We call a diffeomorphism $f : U \to U'$, between D-open subsets $U \subseteq X$ and $U' \subseteq Y$, a *transition* from X to Y. We sometimes write $f : X \dashrightarrow Y$.

We emphasize that local diffeomorphisms are globally defined, and transitions are locally defined.

A diffeology also passes to quotients:

Definition 2.7 (Quotient diffeology)**.** Given a diffeological space X, and equivalence relation $\mathcal{R}$, with quotient map $\pi : X \to X/\mathcal{R}$, the *quotient diffeology* on $X/\mathcal{R}$ consists of those parametrizations $p : \mathsf{U} \to X/\mathcal{R}$ such that about each $r \in \mathsf{U}$ there is an open $\mathsf{V} \subseteq \mathsf{U}$ and a plot $q : \mathsf{V} \to X$ such that $p|_{\mathsf{V}} = \pi \circ q$. In a diagram,

$$\begin{tikzcd} & & X \arrow[d, "\pi"] \\ r \in \mathsf{V} \arrow[urr, "\exists q"] \arrow[r, hook] & \mathsf{U} \arrow[r, "p"] & X/\mathcal{R}. \end{tikzcd}$$

The D-topology of the quotient diffeology is the quotient topology induced by the D-topology on X. Closely related to the quotient diffeology is the notion of a *subduction* between diffeological spaces.

Definition 2.8. A smooth surjective map $f : X \to Y$ is a *subduction* if for every plot $p : \mathsf{U} \to Y$, and every $r \in \mathsf{U}$, there is a neighbourhood V of r and a plot $q : \mathsf{V} \to X$ such that $p|_{\mathsf{V}} = f \circ q$.

Evidently, quotient maps are subductions. Surjective submersions between classical manifolds are also subductions. Subductions satisfy the following universal property ([IZ13, Article 1.51]).

Lemma 2.9. *Suppose $\pi : X \to Y$ is a subduction. Then $f : Y \to Y'$ is smooth (resp. a subduction) if and only if $f \circ \pi : X \to Y'$ is smooth (resp. a subduction).*

Finally, function spaces are naturally diffeological spaces.

Definition 2.10 (Functional diffeology)**.** Let X and Y be diffeological spaces. The *standard functional diffeology* on $C^\infty_{\mathrm{loc}}(X, Y)$ consists of those parametrizations $p : \mathsf{U} \to C^\infty_{\mathrm{loc}}(X, Y)$ satisfying: about each $r_0 \in \mathsf{U}$ and $x_0 \in \operatorname{dom} p(r_0)$, there are open neighbourhoods $sfV \subseteq \mathsf{U}$, and (D-open) $U \subseteq \operatorname{dom} p(r_0)$ such that

- $U \subseteq \operatorname{dom} p(r)$ for all $r \in \mathsf{V}$;
- the map $\mathsf{V} \times U \to Y$ given by $(r, x) \mapsto p(r)(x)$ is smooth.

We equip $C^\infty(X, Y)$ with its subset diffeology inherited from $C^\infty_{\mathrm{loc}}(X, Y)$. This is the coarsest diffeology in which the evaluation map $X \times C^\infty(X, Y) \to Y$ given by $(x, f) \mapsto f(x)$ is smooth. We similarly equip $\mathrm{Diff}_{\mathrm{loc}}(X, Y)$ and $\mathrm{Diff}(X, Y)$ with their subset diffeologies.

For a classical manifold M, one can prove the composition and inversion maps on $\mathrm{Diff}(M)$ are smooth, using the inverse function theorem. This makes $\mathrm{Diff}(M)$ a *diffeological group*, a definition due to Souriau. These groups partly motivated Souriau's definition of diffeology. In general, the composition on $\mathrm{Diff}(X)$ is smooth, but it is an open question whether inversion is necessarily smooth.

Diffeological forms

Now we introduce diffeological differential forms. These will be most useful in Chapter 4

Definition 2.11 (Diffeological forms)**.** A *diffeological k-form* α on X is an assignment to each plot $p : \mathsf{U} \to X$ a differential k-form $\alpha(p) \in \Omega^k(\mathsf{U})$ such that for every open subset V of a Cartesian space, and every smooth map $F : \mathsf{V} \to \mathsf{U}$, we have

$$\alpha(p \circ F) = F^*(\alpha(p)).$$

Denote the set of diffeological k-forms by $\Omega^k(X)$, and the set of diffeological forms by $\Omega^\bullet(X)$.

As with usual differential forms, we can pullback diffeological forms by smooth functions.

Definition 2.12 (Pullbacks)**.** Let $f : X \to Y$ be smooth, and $\alpha \in \Omega^k(Y)$. The *pullback* $f^*\alpha \in \Omega^k(X)$ is defined on the plots of X by $(f^*\alpha)(p) := \alpha(f \circ p)$.

The set $\Omega^k(X)$ is naturally a vector space: given $\alpha, \beta \in \Omega^k(X)$ and $\lambda \in \mathbb{R}$, define for all plots p,

$$(\lambda\alpha + \beta)(p) := \lambda\alpha(p) + \beta(p).$$

The space $\Omega^\bullet(X)$ also carries a differential d and wedge product $\wedge$, respectively defined by

$$(d\alpha)(p) := d\alpha(p), \quad (\alpha \wedge \beta)(p) := \alpha(p) \wedge \beta(p).$$

With respect to the grading $\Omega^\bullet(X) = \bigoplus_{k=0}^\infty \Omega^k(X)$, the space $(\Omega^\bullet(X), d, \wedge)$ is a differential commutative graded algebra. Pullback by a smooth function is a morphism of commutative differential graded algebras.

For example, consider a classical manifold M. To each classical differential form $\underline{\alpha}$ we may associate a diffeological form α by $\alpha(p) := p^*\underline{\alpha}$. Conversely, to each diffeological form α, we can specify a classical form $\underline{\alpha}$ by declaring that in each chart φ of M, we have $(\varphi^{-1})^*\underline{\alpha} := \alpha(\varphi^{-1})$.

Identifying α and $\underline{\alpha}$ identifies the classical de Rham complex on M with the diffeological one. Under this identification, the pullback by a smooth function viewed in the classical and diffeological senses agree. Therefore, we will freely switch between viewing classical forms on M as diffeological ones, and vice versa.

2.2 THE BASICS OF LIE GROUPOIDS

In this section, we introduce basic facts from the study of Lie groupoids. We work inside the bicategory Lie groupoids, denoted **Bi** (after Lerman [Ler10]), whose objects are Lie groupoids, whose arrows are principal bibundles, and whose 2-arrows are bibundle isomorphisms. For a thorough treatment, see Lerman's review [Ler10], or Moerdijk and Mrcŭn's book [MM03]. While the material in this subsection is standard, we take a less common local-first approach to the definition of principal bibundles. Our definition of locally invertible bibundle is new. At the end of this subsection, we introduce the functor $\mathbf{F} : \mathbf{Bi} \to \mathbf{Diffeol}$.

A *Lie groupoid* is a small category $G \rightrightarrows G_0$ with invertible arrows, such that the base G_0 is a (Hausdorff and second-countable) smooth manifold, the arrow space G is a not necessarily Hausdorff nor second-countable smooth manifold, all structure maps are smooth, and furthermore the source map s (hence the target map t) is a smooth submersion with Hausdorff fibers. We write the composition of arrows $g : x \mapsto y$ and $g' : y \mapsto z$ as $g'g : x \mapsto z$.

If $f : M \to G_0$ is a smooth map, we may form the pullback (not necessarily Lie) groupoid f^*G, whose base is M and whose arrows from x to y are arrows from $f(x)$ to $f(y)$. When the map

$$t \circ \mathrm{pr}_1 : G \,{}_s\!\times_f M \to G_0$$

is a submersion, the pullback f^*G is naturally a Lie groupoid. In particular, if $\iota : \mathcal{U} \to G_0$ is a submersion, we may form the pullback groupoid $\iota^*G \rightrightarrows \mathcal{U}$. If U is an open subset of G_0, we denote the corresponding pullback groupoid by $G|_U$, and identify its arrow space with $s^{-1}(U) \cap t^{-1}(U)$. We say that G is *étale* if s (hence t) is a local diffeomorphism. We say that G is *proper* if its arrow space is Hausdorff and the map $(s,t) : G \to G_0 \times G_0$ is proper. By an *isomorphism* of Lie groupoids we mean a smooth functor $F : G \to H$ such that there exists a smooth functor $F^{-1} : H \to G$ for which FF^{-1} and $F^{-1}F$ are the identity functors. The *orbit space* of a Lie groupoid G is the quotient of G_0 by this relation: $x \sim y$ if there is an arrow $x \mapsto y$. We denote the orbit space of G by $|G|$ or G_0/G.

The full definition of a bicategory is rather technical, and we refer the reader to [Ler10] for an introduction. For our purpose, a *bicategory* consists of objects, arrows between objects, and 2-arrows between arrows. These must satisfy some compatibility and composition conditions, which we do not write. The objects and arrows almost form a category, except that composition of arrows need not be associative. Instead, for any composable f, g, h, there exists an invertible 2-arrow $\alpha : f(gh) \to (fg)h$. The arrows and 2-arrows form a category.

A first example of a Lie groupoid is an action groupoid. Given a Lie group G action on a manifold M (from the left), the *action groupoid* $G \ltimes M$ has arrow space $G \times M$ and base space M. The source and target maps are $s(g, x) = x$ and $t(g, x) = g \cdot x$.

Definition 2.13 (Right actions)**.** A *right action* of a Lie groupoid $H \rightrightarrows H_0$ on a manifold P consists of maps $\mu : P \,{}_a\!\times_t H \to P$ and $a : P \to H_0$. We call a the *anchor*, and μ the *multiplication*, and denote $\mu(p, h)$ by $p \cdot h$. We have the following Cartesian square on the left, and require the following commuting diagram of smooth functions on the right

$$\begin{array}{ccc} P \,{}_a\!\times_t H & \xrightarrow{\mathrm{pr}_2} & H \\ \downarrow{\scriptstyle \mathrm{pr}_1} & & \downarrow{\scriptstyle t} \\ P & \xrightarrow{a} & H_0 \end{array} \qquad \begin{array}{ccc} P \,{}_a\!\times_t H & \xrightarrow{\mathrm{pr}_2} & H \\ \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle s} \\ P & \xrightarrow{a} & H_0. \end{array}$$

Furthermore we require

- $(p \cdot h) \cdot h' = p \cdot (hh')$ whenever this makes sense, and
- $p \cdot 1_{a(p)} = p$ for all $p \in P$.

Given a right H action, we may form the *action groupoid* $P \rtimes H$, with arrows $P \,{}_a\!\times_t H$, base P, source μ, target pr_1, and multiplication of arrows $(p, h)(p', h') = (p, hh')$. Furthermore, a and pr_2 assemble into a smooth functor of Lie groupoids $P \rtimes H \to H$.

Remark 2.14.

(i) If $\mathcal{O} \subseteq P$ is an open, H-invariant subset of P, then H acts on $\mathcal{O}$, with the same anchor $a : \mathcal{O} \to H_0$ and the same multiplication $\mu : \mathcal{O} \,{}_a\!\times_t H \to \mathcal{O}$.

(ii) If $V \subseteq H_0$ is open, then $a^{-1}(V)$ need not be H-invariant, but we still have an action of $H|_V$ on $a^{-1}(V)$, with the same anchor $a : a^{-1}(V) \to V$, and the same multiplication.

Fix a Lie groupoid H. A principal H-bundle over a manifold B will consist of a manifold P, a right H action on P, and a map $\pi : P \to B$, satisfying certain axioms. We will denote such a principal bundle by $P \xrightarrow{\pi} B$. We will formulate these axioms using the notion of a trivial bundle.

Definition 2.15. Fix a smooth manifold M, and a map $\phi : M \to H_0$. The *trivial H-bundle over ϕ* consists of the space $P := M\,{}_{\phi}{\times}_t\,H$, equipped with the right H action on $M\,{}_{\phi}{\times}_t\,H$ that is given by the anchor map $s \circ \mathrm{pr}_2 : M\,{}_{\phi}{\times}_t\,H \to H_0$ and the multiplication $(m,h) \cdot h' := (m, hh')$, and equipped with the projection $M\,{}_{\phi}{\times}_t\,H \xrightarrow{\mathrm{pr}_1} M$.

Note that in a trivial H-bundle, the fibers of the projection map $P \xrightarrow{\pi} M$ are the H-orbits, and the H action on P is free, meaning that if $p \cdot h = p \cdot h'$, then $h = h'$. The following example motivates our definition.

Example 2.16. If we take $M = H_0$, and $\phi : H_0 \to H_0$ the identity map, identifying $P = H_0\,{}_{\phi}{\times}_t\,H$ with the arrow space H, we find that the trivial H-bundle over the identity map is the right action of H on its arrow space along the anchor s, given by right multiplication. The trivial H-bundle over an arbitrary $\phi : M \to H_0$ gives a Cartesian square,

$$\begin{tikzcd} M\,{}_{\phi}{\times}_t\,H \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"] & H \arrow[d, "t"] \\ M \arrow[r, "\phi"] & H_0 \end{tikzcd}$$

in which the top map is H-equivariant, and the vertical maps are H-invariant. If we take H to be a Lie group, viewed as a Lie groupoid over a point $H \rightrightarrows \{pt\}$, then the right action of H on itself is the usual one given by right multiplication, and the trivial H-bundle over $M \to \{pt\}$ is a usual trivial H-bundle $M \times H \to M$.

Definition 2.17 (Principal bundles)**.** Fix a smooth manifold B and a Lie groupoid H. A *principal H-bundle over B* consists of a manifold P, a right H action on P along the anchor a, and an H-invariant map $P \xrightarrow{\pi} B$, such that the following holds: about every $b \in B$, there exist an open neighbourhood U, a section $\sigma : U \to P$ of π, and an H-equivariant diffeomorphism Φ fitting in the commutative diagram[2]

2 Here the action of H on $P|_U := \pi^{-1}(U) \subseteq P$ is induced by H acting on P, according to Remark 2.14 (i); the set $P|_U$ is H-invariant by our assumption on π.

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, "\Phi"] \arrow[d, "\pi"] & U \,{}_{a\sigma}\!\times_t H \arrow[dl, "\mathrm{pr}_1"] \\ U. & \end{tikzcd} \tag{2.1}$$

In other words, $P|_U := \pi^{-1}(U)$ is isomorphic to the trivial H-bundle over $a\sigma : U \to H_0$.

Remark 2.18. Other authors, such as [Ler10] and [MM03], define a principal right H-bundle to be a manifold P with right H action, and an H-invariant surjective submersion $P \xrightarrow{\pi} B$, such the map

$$P \,{}_a\!\times_t H \to P \,{}_\pi\!\times_\pi P, \quad (p,h) \mapsto (p, p \cdot h)$$

is a diffeomorphism. This definition is equivalent to ours, and will be useful in Lemma 2.29. Moreover, if we view $P \,{}_\pi\!\times_\pi P$ as a Lie groupoid over P (with source and target the projection maps), the map above assembles with the identity $P \to P$ into an isomorphism of the Lie groupoids $P \rtimes H$ and $P \,{}_\pi\!\times_\pi P$. This observation is relevant in proving Proposition 4.8.

Example 2.19. Every trivial H-bundle over $\phi : B \to H_0$ is a principal H-bundle: take $\sigma : B \to B \,{}_\phi\!\times_t H$ to be $\sigma(b) := (b, 1_{\phi(b)})$, note that $a\sigma = \phi$, and take Φ to be the identity map on $B \,{}_\phi\!\times_t H$. In particular, $H \xrightarrow{t} H_0$, where H acts on its arrows by right multiplication, is a principal H-bundle, with $\sigma(y) = 1_y$. If H is a Lie group viewed as a Lie groupoid over a point, then a principal H-bundle agrees with the usual notion of a principal H group bundle.

We can pull back principal H-bundles by smooth maps. We will need this construction to associate a principal bibundle to a smooth functor.

Definition 2.20 (Pullback bundle)**.** The *pullback* of a principal right H-bundle $P \xrightarrow{\pi} B$ by a smooth map $f : M \to B$, denoted $f^*(P \xrightarrow{\pi} B)$, is the principal right H-bundle consisting of the manifold $M \,{}_f\!\times_\pi P$, the right H action along the anchor map $a \circ \mathrm{pr}_2$ given by $(m,p) \cdot h := (m, p \cdot h)$, and the projection $M \,{}_f\!\times_\pi P \xrightarrow{\mathrm{pr}_1} M$.

For a Lie groupoid G, we can similarly define left G actions, and left principal G-bundles.

Example 2.21. If H is a Lie groupoid acting on P from the right along anchor $a : P \to H_0$, we can get a left action of H on P along a by setting $h \cdot p := p \cdot h^{-1}$. We can similarly get a left action from a right action.

Now we introduce bibundles, principal bibundles, invertible bibundles, and locally invertible bibundles.

Definition 2.22 (Principal bibundle)**.**

- Given Lie groupoids G and H, a *bibundle* $P : G \to H$ is a manifold P equipped with a left G action along a, and a right H action along a', such that a is H-invariant, a' is G-invariant, and the actions of G and H commute.
- We say P is a *principal bibundle* if $P \xrightarrow{a} G_0$ is a principal right H-bundle.
- A bibundle $P : G \to H$ is *invertible* or *biprincipal* if both a and a' are principal bundles. We say Lie groupoids G and H are *Morita equivalent* if there is an invertible bibundle between them.

A *bibundle morphism* from a bibundle P to a bibundle Q is a smooth map $\alpha : P \to Q$ that commutes with the structure maps and is equivariant with respect to both the G and H actions.

Remark 2.23. Every principal bibundle morphism is a bibundle isomorphism. For details, see [MM03, Remark 5.34(5)] or [Ler10, Remark 3.32].

The diagram below depicts a bibundle.

$$\begin{tikzcd} G \circlearrowright & P \arrow[dl, "a"'] \arrow[dr, "a'"] & \circlearrowleft H \\ G_0 & & H_0. \end{tikzcd}$$

If $P : G \to H$ is a bibundle, then after swapping the actions (as in Example 2.21), we obtain a bibundle $P^{-1} : H \to G$. If $P \xrightarrow{a} G_0$ is a principal right H-bundle, $P \xrightarrow{a} G_0$ becomes left H-principal. Similarly, a principal left G-bundle $P \xrightarrow{a'} H_0$ becomes a principal right G-bundle. It follows that if P is invertible, then swapping both the actions returns an invertible bibundle $P^{-1} : H \to G$. The compositions (see the paragraph after Example 2.24) $P^{-1}P$ and PP^{-1} are isomorphic to id_G and id_H, respectively.

Example 2.24. The identity bibundle from a Lie groupoid G to itself, $\mathrm{id}_G : G \to G$, is given by $P = G$, with the left and right actions of G on itself. It is invertible.

For a more sophisticated example, consider a left Lie group action $G \circlearrowright M$ and a right Lie group action $N \circlearrowleft H$. A bibundle $P : G \ltimes M \to N \rtimes H$ between the action groupoids is given by the diagram

$$\begin{tikzcd} & G \circlearrowright P \circlearrowleft H \arrow[dl, "a"'] \arrow[dr, "a'"] & \\ G \circlearrowright M & & N \circlearrowleft H, \end{tikzcd}$$

where a is H-invariant and G-equivariant, and a' is G-invariant and H-equivariant, and the G and H actions on P commute. This bibundle is principal if and only if a is a principal right H-bundle, and it is invertible if and only if, additionally, a' is a principal left G-bundle.

Principal bibundles can be composed: for principal bibundles $P : G \to H$ and $Q : H \to K$, with left and right anchors a, a' and b, b', respectively, the composition $Q \circ P$ is $(P \times_{H_0} Q)/H$ (where the H action on the fibered product $P \times_{H_0} Q$ is $(p, q) \cdot h = (p \cdot h, h^{-1} \cdot q)$), with left and right anchors $\alpha([p, q]) = a(p)$ and $\beta'([p, q]) = b'(q)$. In a diagram,

$$\begin{tikzcd}
 & & (P \times_{H_0} Q)/H \arrow[ddll, bend right, "\alpha"'] \arrow[ddrr, bend left, "\beta'"] & & \\
 & P \arrow[dl, "a"'] \arrow[dr, "a'"] & & Q \arrow[dl, "b"'] \arrow[dr, "b'"] & \\
G_0 & & H_0 & & K_0
\end{tikzcd}$$

See [Ler10, Remark 3.30] for details. However, composition of principal bibundles is not associative. Instead, it is associative up to bibundle isomorphism, so we have a bicategory:

Definition 2.25 (Bicategory of Lie groupoids)**.** The bicategory **Bi** has objects Lie groupoids, arrows principal bibundles, and 2-arrows morphisms of bibundles.

Remark 2.26. By identifying isomorphic principal bibundles, we obtain the Hilsum-Skandalis category **HS**. Its objects are Lie groupoids, and its arrows are isomorphism classes of principal bibundles.

Example 2.27. Every smooth functor of Lie groupoids induces a principal bibundle. Indeed, given a smooth functor $F : G \to H$ with $F_0 : G_0 \to H_0$, the principal H-bundle $F_0^*(H \xrightarrow{t} H_0)$ is naturally a bibundle $G \to H$, which we denote $\langle F \rangle$. The groupoid G acts along the anchor map pr_1, with multiplication $g \cdot (x, h) := (t(g), F(g)h)$. The map $F \mapsto \langle F \rangle$ respects composition in the sense that $\langle F \circ G \rangle \cong \langle F \rangle \circ \langle G \rangle$.

A smooth natural transformation between smooth functors $\alpha : F \to G$ gives rise to a bibundle morphism $\langle \alpha \rangle : \langle F \rangle \to \langle G \rangle$. This bibundle morphism is an isomorphism by Remark 2.23.

We now introduce the notion of the restriction of a bibundle, and then, of a locally invertible bibundle.

Definition 2.28. If $P : G \to H$ is a bibundle as in Definition 2.22 and $U \subseteq G_0$ and $V \subseteq H_0$ are open subsets then $P|_U^V := a^{-1}(U) \cap (a')^{-1}(V)$, equipped with the actions of $G|_U$ and $H|_V$ described by Remark 2.14, is a bibundle, which we call the *restriction* of P.

Lemma 2.29. *If G and H are Lie groupoids, $P : G \to H$ is a principal bibundle, and $U \subseteq G_0$ and $V \subseteq H_0$ are open subsets, and $Q : G|_U \to H|_V$ is a principal bibundle such that the diagram below commutes (up to isomorphism of bibundles)*

$$\begin{tikzcd} G|_U \arrow[r, "Q"] \arrow[d, "\langle \iota_U \rangle"] & H|_V \arrow[d, "\langle \iota_V \rangle"] \\ G \arrow[r, "P"] & H, \end{tikzcd} \tag{2.2}$$

then $P|_U^V$ exists and is isomorphic to Q (here ι_U is the inclusion functor). Conversely, if $P|_U^V$ is principal, then the diagram (2.2) commutes with $Q := P|_U^V$.

Proof. By definition, the principal bibundle $\langle \iota_U \rangle$ is $U \,{}_{\iota_U}\!\times_t G$, which we identify with $t^{-1}(U)$ via $(x, g) \mapsto g$. The $G|_U$ action is by left multiplication. Then

$$P \circ \langle \iota|_U \rangle = (t^{-1}(U) \,{}_s\!\times_a P)/G$$

We denote its elements by $[g, p]$. Similarly, we identify the principal bibundle $\langle \iota_V \rangle$ with $t^{-1}(V)$, and find that

$$\langle \iota_V \rangle \circ Q = (Q \,{}_{b'}\!\times_t t^{-1}(V))/H|_V,$$

where b' is the right anchor for Q. We denote its elements by $[q, h]$.

Denote an isomorphism $P \circ \langle \iota_U \rangle \leftrightarrow \langle \iota_V \rangle \circ Q$ element-wise by $[g, p] \leftrightarrow [q, h]$. Then, define a map $Q \to P$ by $q \mapsto g \cdot p$, where $[q, 1_{b'(q)}] \leftrightarrow [g, p]$ under the assumed isomorphism. This is well-defined because $[g_1, p_1] = [g_2, p_2]$ implies $g_1 \cdot p_1 = g_2 \cdot p_2$. Now, notice that $a(g \cdot p) = t(g) \in U$, and

$$a'(g \cdot p) = a'(p) = b'(q) \in V,$$

where the last equality follows from the requirement that an isomorphism of bibundles commutes with the anchor maps. So we have defined a map $Q \to P|_U^V$. Its inverse is the map $p \mapsto q \cdot h$, where $[1_{a(p)}, p] \leftrightarrow [q, h]$. Both these maps are smooth, and furthermore they are $G|_U \times H|_V$-equivariant. This shows that $P|_U^V$ is a principal bibundle, and is isomorphic to Q.

For the converse, we define the diffeomorphism $P \circ \langle \iota_U \rangle \to \langle \iota_V \rangle \circ P|_U^V$ as follows. Given $[g, p]$, denote $p_* := g \cdot p$, and note that $a(p_*) = t(g) \in U$. Now take a local section $\sigma : U \to P|_U^V$ of a about $a(p_*)$, which is possible because $P|_U^V \xrightarrow{a} U$ is principal by assumption. Then $q := \sigma(a(p_*))$ and p_* are in the same a-fiber, hence, by principality of the H action, there is a unique $h \in H$ (depending smoothly on p and g; see Remark 2.18) such that $q \cdot h = p_*$. Necessarily $t(h) = a'(q) \in V$. We claim the desired diffeomorphism is $[g, p] \mapsto [q, h]$.

We show this is well-defined. Suppose $[g_1, p_1] = [g_2, p_2]$. Then, as above, we have $g_1 \cdot p_1 = g_2 \cdot p_2 =: p_*$, so we only need to show the definition is independent of the section σ. Suppose $\sigma_i : U \to P|_U^V$ are two local sections of a about $a(p_*)$. Set $q_i := \sigma_i(a(p_*))$, and set $h_i \in H$ to be the unique arrows such that $q_i \cdot h_i = p_*$. Then both $q_i \cdot h_i$ are in the same a-fiber, so there is some unique $\tilde{h} \in H$ such that $q_2 = q_1 \cdot \tilde{h}$. Then $q_2 \cdot h_2 = q_2 \cdot (\tilde{h} h_2)$, and uniqueness of h_1 means $\tilde{h} h_2 = h_1$. It follows that $[q_2, h_2] = [q_1 \cdot \tilde{h}, \tilde{h}^{-1} h_1] = [q_1, h_1]$, as required.

We leave it to the reader to check that this map is $G|_U \times H$-equivariant. It is also smooth: the maps $(g, p) \mapsto (\sigma(a(p_*)), h)$ provide local liftings. Finally, it is a bibundle isomorphism by Remark 2.23. □

Definition 2.30 (Locally invertible bibundles)**.** A bibundle $P : G \to H$ is *locally invertible* if it is principal, and about every $x \in G_0$ and $y \in a'(a^{-1}(x)) \subseteq H_0$, there are open neighbourhoods U of x and V of y, such that the restriction $P|_U^V : G|_U \to H|_V$ is invertible.

The composition of invertible bibundles is invertible, and the composition of locally invertible bibundles is locally invertible, by Lemma 2.29. Our main source of locally invertible bibundles comes from certain smooth functors $F : G \to H$.

Lemma 2.31. *Suppose $F : G \to H$ is a smooth functor such that, for every $x \in G_0$, there exists open neighourhoods U of x in G_0 and V of $F(x)$ in H_0 such that the functor F restricts to an isomorphism $F|_U^V : G|_U \to H|_V$. Then $\langle F \rangle$ is a locally invertible bibundle.*

Proof. Take $F : G \to H$ as given in the setup. We will show that $\langle F \rangle|_V^U$ is invertible. First, observe that setting $Q := \langle F|_U^V \rangle$ and $P := \langle F \rangle$ makes diagram (2.2) commute, so by Lemma 2.29, we see that $\langle F|_U^V \rangle$ is isomorphic to $\langle F \rangle|_U^V$. Therefore it suffices to show $\langle F|_V^U \rangle$ is invertible.

Since $F|_U^V$ is an isomorphism, we simplify notation and assume F is an isomorphism, and prove $\langle F \rangle$ is invertible. In other words, we must show $G_0 \; {}_F\times_t H \xrightarrow{s \circ \mathrm{pr}_2} H_0$ is a left G-principal bundle. In fact, we show it is isomorphic to a trivial left G-bundle. Take the section σ of $s \circ \mathrm{pr}_2$ defined by $\sigma(y) := (F^{-1}(y), 1_y)$, and define Φ, in the diagram below

$$\begin{tikzcd} G \; {}_s\times_{\mathrm{pr}_1 \sigma} H_0 \arrow[r, "\Phi"] \arrow[dr, "\mathrm{pr}_2"'] & G_0 \; {}_F\times_t H \arrow[d, "s\,\mathrm{pr}_2"] \\ & H_0 \end{tikzcd}$$

by

$$\Phi(g, y) := (t(g), F(g)), \quad \Phi^{-1}(x, h) = (F^{-1}(h), s(h)).$$

Unwinding the definitions yields that these are inverses of each other, and that they are G-equivariant, is a matter of unwinding the definitions. □

We end this subsection by describing the quotient functor $\mathbf{F} : \mathbf{Bi} \to \mathbf{Diffeol}$. This functor, and parts of the following proposition, have appeared elsewhere in the literature, for instance in [Wan17, Lemma 6.5]. We give the entire argument for completeness; the clause concerning locally invertible bibundles is new.

Proposition 2.32. *Suppose $P : G \to H$ is a principal bibundle. There is a unique map $|P| : |G| \to |H|$ for which the diagram below commutes*

$$\begin{tikzcd}[column sep=small] & P \arrow[dl, "a"'] \arrow[dr, "a'"] & \\ G_0 \arrow[d, "\pi"'] & & H_0 \arrow[d, "\pi'"] \\ |G| \arrow[rr, "|P|"] & & |H|. \end{tikzcd} \tag{2.3}$$

The map $|P|$ is diffeologically smooth. If $\mathrm{id}_G : G \to G$ is the identity bibundle, then $|\mathrm{id}_G| = \mathrm{id}_{|G|}$. If $Q : H \to K$ is another principal bibundle, then $|Q \circ P| = |Q| \circ |P|$. If $P, P' : G \to H$ are isomorphic, then $|P| = |P'|$. Hence $\mathbf{F} : \mathbf{Bi} \to \mathbf{Diffeol}$, which takes an object $G \rightrightarrows G_0$ to $|G|$, takes an arrow P to $|P|$, and takes a 2-arrow $\alpha : P \to Q$ to $1_{|P|}$, is a well-defined functor of bicategories.

If P is invertible, then $|P|$ is a diffeomorphism. If P is locally invertible, then $|P|$ is a local diffeomorphism.

Proof. Because the given diffeology on $|G|$, which is the quotient diffeology induced by π, coincides with the quotient diffeology that is induced by πa, if $|P|$ is well-defined, then $|P|\pi a = \pi' a'$ implies that $|P|$ is smooth and uniquely determined by a and a': necessarily, $|P|$ is given by

$$[x] \mapsto [a'(p)], \text{ where } a(p) = x.$$

We now check this is well-defined. Suppose $g : x \mapsto y$, and $a(p) = x$ and $a(q) = y$. We find an arrow in H taking $a'(q)$ to $a'(p)$. First, $s(g) = x = a(p)$, so $g \cdot p$ is well-defined. Applying a gives

$$a(g \cdot p) = t(g) = y = a(q).$$

Because $P \xrightarrow{a} G_0$ is H principal, we can find (unique) arrow h such that $(g \cdot p) \cdot h = q$. The fact the action is well-defined means $t(h) = a'(g \cdot p)$, and this is $a'(p)$ by G-invariance of a'. On the other hand,

$$a'(q) = a'(g \cdot p \cdot h) = s(h),$$

so h is the desired arrow.

Now, suppose $P : G \to H$ and $Q : H \to K$ are principal bibundles, with left and right anchors a, a' and b, b', respectively. Recall that composition $Q \circ P$ is given by the diagram

$$\begin{tikzcd}
 & & P \times_{H_0} Q/H \arrow[ddll, bend right, "\alpha"'] \arrow[ddrr, bend left, "\beta'"] & & \\
 & P \arrow[dl, "a"'] \arrow[dr, "a'"] & & Q \arrow[dl, "b"'] \arrow[dr, "b'"] & \\
G_0 \arrow[d] & & H_0 \arrow[d] & & K_0 \arrow[d] \\
|G| \arrow[rr, "|P|"] \arrow[rrrr, dashed, bend right, "|Q \circ P|"'] & & |H| \arrow[rr, "|Q|"] & & |K|
\end{tikzcd}$$

Choose $x \in G_0$, then choose $p \in P$ such that $a(p) = x$, then choose q such that $b(q) = a'(p)$. Then

$$|Q| \circ |P|([x]) = Q([a'(p)]) = [b'(q)]$$

But we can now pick $[p, q] \in Q \circ P$, where $\alpha([p, q]) = x$ and $\beta'([p, q]) = b'(q)$. Then $|Q \circ P|([x]) = [b'(q)]$ too, as required.

Suppose $\alpha : P \to Q$ is an isomorphism of bibundles. Fix $a(p) = x$, so that $|P|([x]) = [a'(p)]$. Then $b(\alpha(p)) = x$, so $|Q|([x]) = [b'(\alpha(p))]$. But $b'\alpha = a'$, so $|Q|([x]) = [a'(p)] = |P|([x])$.

If P is invertible, the inverse of $|P|$ is $|P^{-1}|$. If P is locally invertible, for $x \in G_0$ and $y \in a'(a^{-1}(x))$, take neighbourhoods U of x and V of y such that $P|_U^V$ is invertible. By Lemma 2.29 we have the diagram (2.2), and passing to the quotient yields the diagram

$$\begin{tikzcd}
|U| \arrow[r, "|P|_U^V|"] \arrow[d] & |V| \arrow[d] \\
|G| \arrow[r, "|P|"] & |H|.
\end{tikzcd}$$

See Lemma 5.16 for the identification of $|U| = U/G|_U$ with an open subset of $|G|$. The vertical arrows are inclusions, and therefore the top arrow is $|P|$ restricted to a map $|U| \to |V|$. But the top arrow is also a diffeomorphism because $P|_U^V$ is invertible, so we conclude $|P|$ is a local diffeomorphism. □

Étale Lie groupoids

An *étale* Lie groupoid $G \rightrightarrows G_0$ is one which has discrete isotropy groups. Equivalently, $\dim G = \dim G_0$. We now define effective étale groupoids. We will use pseudogroups.

Definition 2.33. A *pseudogroup* Ψ on a manifold M is a set of transitions $M \dashrightarrow M$ (see Definition 2.6) that contains the identity, is closed under composition, inversion, and restriction to open subsets, and satisfies the following locality axiom:

> If ψ is a transition on M and there exists a cover of its domain for which $v|_U \in \Psi$ for every element U of the cover, then $\psi \in \Psi$.

A pseudogroup Ψ on M partitions M into *orbits* $\{f(x) \mid \psi \in \Psi\}$. The intersection of pseudogroups is a pseudogroup. For a set Ψ_0 of transitions, we call the intersection of all pseudogroups containing Ψ_0 the pseudogroup *generated* by Ψ_0.

Remark 2.34. A pseudogroup Ψ on M is generated by a set of transitions Ψ_0 if and only if every $\psi \in \Psi$ is locally a composition of elements of ψ_0 and their inverses. Precisely, for every $x \in \operatorname{dom}\psi$, there are $\psi_i \in \Psi_0$, and $\epsilon_i \in \{1, -1\}$, such that $\operatorname{germ}_x \psi = \operatorname{germ}_x \psi_1^{\epsilon_1} \circ \cdots \circ \psi_n^{\epsilon_n}$.

Example 2.35.

- The collection of all transitions from a manifold M to itself is a pseudogroup, called the *Haefliger pseudogroup*.
- Given an étale Lie groupoid $G \rightrightarrows G_0$, the collection of *local bisections* of G, defined as $\{t \circ \sigma \mid \sigma \text{ is a local inverse of } s\}$, generates a pseudogroup on G_0, which we denote $\Psi(G)$.

Remark 2.36. Conversely, given a pseudogroup Ψ on M, we form its *germ groupoid* $\Gamma(\Psi) \rightrightarrows M$. Its arrows are the germs of elements of Ψ. The source map is $s(\operatorname{germ}_x \psi) = x$, the target is $t(\operatorname{germ}_x \psi) = \psi(x)$, the multiplication is $\operatorname{germ}_x \psi \cdot \operatorname{germ}_{x'} \psi' = \operatorname{germ}_{x'} \psi\psi'$, the unit map is $x \mapsto \operatorname{germ}_x \mathrm{id}$, and the inversion is $(\operatorname{germ}_x \psi)^{-1} = \operatorname{germ}_{\psi(x)} \psi^{-1}$. We equip the arrow space $\Gamma(\Psi)$ with the smooth structure given by the following atlas. For each $\psi \in \Psi$ and chart $r : U \to \Omega$ of M with $U \subseteq \operatorname{dom}\psi$, take the chart

$$\{\operatorname{germ}_x \psi \mid \psi(x) \in U\} \to \Omega, \quad \text{given by} \quad \operatorname{germ}_x \psi \mapsto r(x).$$

Now we introduce the effect functor, and we define an effective étale groupoid.

Definition 2.37. Let G be étale Lie groupoid. Let $\Psi(G)$ be its associated pseudogroup (see Example 2.35). Let $\Gamma(\Psi(G))$ be the its germ groupoid. The *effect* functor $\mathrm{Eff} : G \to \Gamma(\Psi(G))$ is identity on G_0, and

$$\mathrm{Eff}(g) := \mathrm{germ}_{s(g)}(t \circ \sigma),$$

where σ a local inverse of s taking $s(g)$ to g. This is a surjective local diffeomorphism on arrows. An étale Lie groupoid G is *effective* if Eff is also injective on arrows. In this case, G is isomorphic to $\mathrm{Eff}(G) = \Gamma(\Psi(G))$.

The germ groupoid $\Gamma(\Psi)$ of a pseudogroup is effective. In general, for an étale Lie groupoid G and for a pseudogroup Ψ, we have the relations

$$\Gamma(\Psi(G)) = \mathrm{Eff}(G), \quad \Psi(\Gamma(\Psi)) = \Psi. \tag{2.4}$$

2.3 THE HOLONOMY GROUPOID OF A REGULAR FOLIATION

Fix a regular foliation $(M, \mathcal{F})$, by which we mean a partition $\mathcal{F}$ of M into connected (weakly-embedded) submanifolds L for which the associated distribution $x \mapsto T_xL$ is smooth and involutive. In this section, we assume familiarity with regular foliations, but note that we cover foliations in greater detail in Chapter 3. A good reference for this section is [MM03]. Associated to $(M, \mathcal{F})$ is a distinguished Lie groupoid, the *holonomy groupoid*.

Definition 2.38 (Holonomy groupoid)**.** The *holonomy groupoid* associated to $\mathcal{F}$ is a Lie groupoid $\mathrm{Hol} \rightrightarrows M$, whose arrows $x \mapsto y$ are the holonomy classes of leafwise paths from x to y (see [MM03], Chapter 2.1, for a definition of holonomy), and multiplication is given by concatenation of paths.

The two key facts we require about Hol, which we explain below, are:

- it is source-connected;
- it is Morita equivalent to an étale Lie groupoid with countably generated pseudogroup.

First, we call a Lie groupoid $G \rightrightarrows G_0$ *source-connected* if the source-fibers are all connected. As with Lie groups, we can consider the *source-connected identity component* $G^\circ \rightrightarrows G_0$ of a Lie groupoid. The arrows in G° are those arrows $g \in G$ such that g and $1_{s(g)}$ belong to the same component of $s^{-1}(s(g))$. This is an open source-connected subgroupoid of G (see [Mei03]). The groupoid Hol is source-connected ([MM03], page 157).

Second, recall that we call a Lie groupoid *étale* if the source map is a local diffeomorphism. To each étale Lie groupoid $G \rightrightarrows G_0$, we associate the pseudogroup $\Psi(G)$ on G_0, given by

$$\Psi(G) := \{t \circ \sigma \mid \sigma \text{ is a local section of } s, \text{ and } t \circ \sigma \text{ is a diffeomorphism}\}.$$

We will work with countably generated pseudogroups.

Definition 2.39 (Countably generated pseudogroups)**.** A pseudogroup Ψ is *countably generated* if it is generated (see Remark 2.34) by a countable set of transitions.

We now phrase our second property of Hol as a lemma, but first need a definition.

Definition 2.40 (Complete transversal)**.** A *complete transversal* to a codimension q regular foliation $\mathcal{F}$ is an immersion from a q-dimensional submanifold $\iota : S \hookrightarrow M$, such that S is transverse to every leaf it meets, and it meets every leaf; i.e. $d\iota_x(T_xS) \oplus T_{\iota(x)}L = T_{\iota(x)}M$, where L is the leaf about $\iota(x)$, and the image of ι meets every leaf at least once.

This differs slightly from the definition in [MM03] (their Example 5.19), because we do not assume ι to be injective. Therefore, ι is not necessarily the inclusion map for a submanifold of M. However, we do not need injectivity of ι for the next lemma. Complete transversals to regular foliations always exist, as we show in the proof below.

Lemma 2.41. *For a regular foliation $\mathcal{F}$ with complete transversal $\iota : S \hookrightarrow M$, we may pull back* Hol *along ι, to get a Lie groupoid* $\mathrm{Hol}_S \rightrightarrows S$*. This is Morita equivalent to* Hol*. It is also étale. Its associated pseudogroup, which we call the* holonomy pseudogroup *of $\mathcal{F}$, is countably generated.*

Proof. Assume $\mathcal{F}$ has codimension q. By [MM03], Chapter 1.2, fact (i), we may take a countable foliation atlas $\{\varphi_i : U_i \to \mathbb{R}^{n-q} \times \mathbb{R}^q\}$, such that the change-of-coordinates diffeomorphisms have the form $\varphi_{ji}(x,y) = (g_{ji}(x,y), h_{ji}(y))$, where h_{ji} is a locally-defined diffeomorphism of $\mathbb{R}^q$. An important consequence of this fact (i) is that we can choose the U_i such that, whenever U_i meets U_j, there is some other foliation chart (not necessarily in our collection), that contains $U_i \cup U_j$. Set $S_i := \varphi_i^{-1}(\{0\} \times \mathbb{R}^q)$, an embedded submanifold of M, and, viewing h_{ji} as locally-defined diffeomorphism on $\{0\} \times \mathbb{R}^q$, identify h_{ji} with the map, also denoted h_{ji},

$$S_i \to S_j, \quad x \mapsto \varphi_i^{-1} h_{ji} \varphi_j(x).$$

We only make this definition if U_i meets U_j. In this case, because there is some foliation chart containing $U_i \cup U_j$, the map h_{ji} is a diffeomorphism $S_i \to S_j$ preserving the plaques of U. Set $S := \bigsqcup S_i$, and let Ψ be the pseudogroup (countably) generated by the h_{ji}. By construction of holonomy ([MM03], Chapter 2.1), given any leafwise (contained in a leaf) path α from $x \in S_i$ to $y \in S_j$, its holonomy $\mathrm{Hol}^{S_j,S_i}(\alpha)$ is an element of Ψ.

Equip S with the natural immersion $\iota : S \to M$. By construction, this is a complete transversal to $\mathcal{F}$. The pullback $\mathrm{Hol}_S \rightrightarrows S$ is Morita equivalent to Hol, and it is étale (see [MM03], page 135, for details). Moreover, we identify the arrows in Hol_S with arrows in Hol from S to S. Suppose $t \circ \sigma \in \Psi(\mathrm{Hol}_S)$. To show $t \circ \sigma \in \Psi$, we may show this is true locally. Fix $x \in \mathrm{dom}(\sigma)$, and say $x \in S_i$. Let α be a representative of $\sigma(x)$, and say α is a leafwise path from x to $y \in S_j$. Then there exists a smooth map $H : [0,1] \times S_i \to S_j$ (perhaps shrinking S_i and S_j) such that $H(\cdot, x')$ is a leafwise path from x' to $\mathrm{Hol}^{S_j,S_i}(\alpha)(x')$ ([MM03], page 118). Then, $x' \mapsto [H(\cdot, x')]$ is a section of s through $[\alpha]$ at x (we use square brackets to denote holonomy classes); because s is a local diffeomorphism, this section is σ. We conclude that, locally at x, $t \circ \sigma = \mathrm{Hol}^{S_j,S_i}(\alpha) \in \Psi$. So then $\Psi(\mathrm{Hol}_S) \subseteq \Psi$. Because Ψ is countably-generated, so is $\Psi(\mathrm{Hol}_S)$. □

Many sources (for example, [Mol88]), define the holonomy pseudogroup to be Ψ above, in which case the holonomy pseudogroup is countably generated almost by definition. However, this obscures the connection of the holonomy pseudogroup to basic forms (which we realize through Propositions 4.7 and 4.8). Furthermore, while the last assertion of Lemma 2.41 is cited without proof in [MM03], Example 5.23 (2), and generally seems to be an accepted fact, we could not find a detailed argument in the literature illuminating why $\Psi(\mathrm{Hol}_S) = \Psi$.

3 SINGULAR FOLIATIONS AND MOLINO EQUIVALENCE

A singular foliation is a partition of a manifold into connected submanifolds, called leaves, of perhaps varying dimension, which fit together smoothly in an appropriate sense. Singular foliations lie beneath many structures one frequently encounters in geometry: the partition of a manifold into orbits of a connected smooth Lie group action, the partition of a Poisson manifold into symplectic leaves, and the partition of a manifold into the maximal integral submanifolds of an involutive distribution are all singular foliations. In the last example, the leaves of the resulting singular foliation all have the same dimension, and thus we obtain a regular foliation. A global version of the Frobenius theorem asserts that every regular foliation arises in this way.

Theorem 3.1 (Global Frobenius)**.** *Let M be a manifold. An involutive distribution Δ has integral submanifolds, and the partition of a manifold M into the maximal integral submanifolds is a regular foliation $\mathcal{F}_\Delta$. Conversely, given a regular foliation $\mathcal{F}$, the associated distribution $T\mathcal{F}$ is involutive. The assignments*

$$\mathcal{F} \mapsto T\mathcal{F}, \quad \Delta \mapsto \mathcal{F}_\Delta$$

are inverses of each other.

We employ the adjective "global" because in his original paper [Fro77], Frobenius worked locally, and did not consider foliations; in modern terms (as opposed to the language of Pfaffian systems used at the time), Frobenius proved that a regular distribution Δ has integral submanifolds if and only if it is involutive. Furthermore, Frobenius was not - and did not claim to be - the first to prove the result that now bears his name. He was building from the work of Jacobi [Jac27], who gave sufficient integrability conditions in a special case, and of Clebsch [Cle66], who expanded Jacobi's conditions the general case. Frobenius also singled out the contributions of Deahna [Dea40], who gave necessary and sufficient conditions for integrability (cf. Remark 3.21). For more on the history of Frobenius's theorem, see the treatments by Hawkins [Haw05] and Samelson [Sam01].

For singular foliations the situation remained unclear for some time. Hermann [Her62] was the first to give some sufficient conditions for integrability in terms of Lie subalgebras of vector fields (stated in Example 3.22). Shortly after, Nagano [Nag66] established Frobenius's theorem holds in the analytic case. Matsuda [Mat68], and Lobry [Lob70] worked to extended the sufficient conditions of Hermann, but like Hermann did not propose integrability conditions on the singular distributions themselves. Then, in 1973-1974, Stefan [Ste74] and Sussmann [Sus73] independently published results that, taken together, successfully generalize the global Frobenius theorem as stated above; the integrability condition on Δ is that it is invariant under the flow of any vector field tangent to Δ. In Section 3.1 of this paper, we proceed carefully toward Stefan and Sussmann's results, highlighting the independent contributions of both authors. For a broader perspective on the history of integrability theorems for singular distributions, see Lavau's survey [Lav18].

Diffeology was introduced in the 1980s by Souriau, and was therefore not available to Stefan and Sussmann as they wrote in the early 1970s. However, Stefan's work in particular uses many tools that arise naturally in diffeology, and he draws conclusions that are best interpreted diffeologically. We emphasize these connections to diffeology throughout Section 3.1, but highlight here Corollary 3.25 and Proposition 3.26, which imply that it is equivalent to consider singular foliations, and orbits of D-connected diffeological groups acting smoothly on M.

In Section 3.2, we contrast two approaches to the transverse geometry of a singular foliation: one equips the leaf space $M/\mathcal{F}$ with the quotient diffeology, and the other defines a notion of transverse equivalence directly. We call this Molino transverse equivalence, after Molino, whose notion of transverse equivalence ([Mol88, Definition 2.1]) for regular foliations this generalizes. By Proposition 3.43, a Molino transverse equivalence always induces a diffeomorphism between the leaf spaces, but the converse does not hold in general; in Example 3.51, we give two regular foliations that are not Molino transverse equivalent, but which have diffeomorphic leaf spaces. It is thus a substantive question whether there is a class of singular foliations for which the diffeology of its leaf space determines its Molino transverse equivalence class. We give one affirmative answer in Proposition 3.53 in terms of quasifold groupoids, which were introduced in [KM22], or Chapter 5 of this thesis. This specializes to orbifolds.

Our Molino transverse equivalence is intimately related to Garmendia and Zambon's Hausdorff Morita equivalence of singular foliations, defined in [GZ19]. We note that Garmendia and Zambon begin with a substantially different notion of singular foliation, which we call an Androulidakis-Skandalis singular foliation, cf. Definition 3.30. They therefore have a different, generally finer, notion of equivalence. Our Molino transverse equivalence may be viewed as an extension of Garmendia and Zambon's Hausdorff Morita equivalence to Stefan singular foliations (Definition 3.6).

Finally, in Section 3.3, we apply our Molino transverse equivalence to the basic cohomology associated of a singular foliation. This cohomology was first introduced by Reinhart [Rei59] in the regular case. Our main result of this section, Corollary 3.59, shows that the basic cohomology is invariant under Molino transverse equivalence. This is consistent with the expectation that the basic cohomology depends only on the transverse geometry of the singular foliation. As a consequence, we see by Proposition 3.62 that if the quotient map $\pi : M \to M/\mathcal{F}$ induces an isomorphism $\pi^* : H^\bullet(M/\mathcal{F}) \to H_b^\bullet(M, \mathcal{F})$, where $H^\bullet(M/\mathcal{F})$ is the cohomology associated to the diffeological differential forms on $M/\mathcal{F}$, the same is true for all singular foliations in the Molino transverse equivalence class of $(M, \mathcal{F})$. Whether π always induces an isomorphism remains an open question, which we address in more detail in Chapter 4

We will assume the facts from diffeology and Lie groupoids included in Chapter 2, but remind the reader that Iglesias-Zemmour's book [IZ13] is an excellent source for the former, and Moerdijk and Mrčun's book [MM03] and Lerman's article [Ler10] are excellent sources for the latter. For a history of the development of singular foliations, see the survey by Lavau [Lav18].

We will reiterate the structure of this article. In Section 3.1, we give a careful description of the definition of a singular foliation, and an overview of the Stefan-Sussmann theorem. We pay special attention to diffeological considerations. In Section 3.2, we introduce Molino transverse equivalence of singular foliations, and relate this to the diffeology of the leaf space, and Morita equivalence of Lie groupoids. Finally, in Section 3.3, we show that the basic cohomology of a singular foliation is invariant under Molino transverse equivalence.

3.1 THE BASICS OF SINGULAR FOLIATIONS

A singular foliation is a partition of a manifold[1] M into connected submanifolds, called *leaves*, fitting together smoothly in an appropriate sense. Before we give a definition of singular foliation, however, we must establish what we mean by "submanifold." First candidates are immersed submanifolds.

1 by "manifold," we mean set equipped with a maximal smooth atlas (smooth structure) whose induced topology is Hausdorff and second-countable. When required, we may denote the smooth structure by a letter such as σ, and refer to the manifold as (M, σ).

Definition 3.2. A subset L of M, together with a manifold structure σ_L, is an *immersed submanifold* of M if the inclusion $\iota : (L, \sigma_L) \hookrightarrow M$ is an injective immersion.

Remark 3.3. The smooth structure σ_L is part of the data of the immersed submanifold (L, σ_L). There may be other smooth structures σ_L' on L for which (L, σ_L') is immersed, but is not diffeomorphic to (L, σ_L). For example, a figure-eight may be immersed in $\mathbb{R}^2$ in two different ways. Note the topology of (L, σ_L) is generally finer than the subspace topology.

Diffeology provides other candidates for submanifolds, namely diffeological and weakly-embedded submanifolds.

Definition 3.4. A subset L of M is a *diffeological submanifold* of M if its subspace diffeology is a manifold diffeology. Diffeological submanifolds always carry the manifold structure given by their subspace diffeology. We call such L a *weakly-embedded submanifold* if the inclusion $i : L \hookrightarrow M$ is an immersion.

Remark 3.5. Weakly-embedded submanifolds have unique smooth structures for which the inclusion is an immersion; they are unambiguously immersed submanifolds. The topology of a diffeological submanifold is finer than the subspace topology. Due to a theorem of Joris [Jor82], the cusp $x^2 = y^3$ is a diffeological submanifold of $\mathbb{R}^2$ that is not weakly-embedded. For a detailed discussion, see [KMW22].

Stefan required the leaves of a singular foliation to be weakly-embedded in his definition from [Ste74].

Definition 3.6. A *(Stefan) singular foliation* of a smooth manifold M is a partition $\mathcal{F}$ of M into connected, weakly-embedded submanifolds L, called *leaves* such that about each $x \in M$, there exists a chart ψ of M for which

(a) ψ is a diffeomorphism $V \to U \times W$, where U and W are open neighbourhoods of the origin in $\mathbb{R}^k$ and $\mathbb{R}^{n-k}$, respectively, and k is the dimension of the leaf through x;

(b) $\psi(x) = (0, 0)$;

(c) If L is any leaf of $\mathcal{F}$, then $\psi(L) = U \times \ell$, for some $\ell \subseteq W$.

Remark 3.7. If every leaf has the same dimension, we call the foliation *regular*. In this case, Stefan's definition coincides with the usual chart-based definition of a foliation.

Remark 3.8. In the next section, we will consider the assignment to each $x \in M$ the subspace $T_xL \subseteq T_xM$. This is an example of a singular distribution, and the leaves L are its maximal integral submanifolds (cf. Definitions 3.13, 3.16, and 3.18).

An attractive feature of this definition is that it is intrinsic to the manifold M and the partition into leaves L (whose smooth structures are determined by the smooth structure on M, see Remark 3.5). This is the first intrinsic definition of singular foliation to appear in the literature. Prior to Stefan, singular foliations were known only through examples appearing, for instance, in control theory.

Example 3.9. Given a Lie group G acting smoothly on a manifold M, the connected components of the G-orbits assemble into a singular foliation. More generally, the connected components of the orbits of a Lie groupoid $G \rightrightarrows M$ form a singular foliation of M. Yet more generally, every Lie algebroid $A \implies M$ induces a singular foliation on M. We treat these examples later in Example 3.22.

The condition that the leaves of $\mathcal{F}$ are weakly-embedded submanifolds seems at first unnecessarily restrictive. *A priori*, we might gain generality by only requiring that the leaves are connected immersed submanifolds of M. Kubarski [Kub90] in 1990 showed that in fact, we gain no generality from this weaker assumption.

Theorem 3.10 (Kubarski)**.** *Suppose M is a smooth manifold, and $\mathcal{F}$ is a partition of M such that*

- *Each $L \in \mathcal{F}$ comes equipped with a smooth structure σ_L for which (L, σ_L) is a connected immersed submanifold of M, and*
- *About each $x \in M$, there is a chart ψ of M satisfying (a) through (c) in Definition 3.6, where the k in condition (a) is the dimension of the leaf (L, σ_L) about x.*

Then each L is a weakly-embedded submanifold of M, and consequently, $\mathcal{F}$ is a Stefan singular foliation.

On the other hand, the notion of a diffeological submanifold provides another potential generalization to Stefan's definition. We show that again, we gain no generality under this weaker assumption.

Proposition 3.11. *Suppose M is a smooth manifold, and $\mathcal{F}$ is a partition of M into connected, diffeological submanifolds such that about each $x \in M$, there exists a chart ψ of M satisfying (a) through (c) in Definition 3.6. Then each $L \in \mathcal{F}$ is a weakly-embedded submanifold of M, and consequently, $\mathcal{F}$ is a Stefan singular foliation.*

Proof. It suffices to show the inclusion $\iota : L \hookrightarrow M$ is an immersion. Take $x \in L$, and fix a chart ψ satisfying (a) through (c) of Definition 3.6. The intersection $L \cap V$ is an open subset of L.[2] Consider the diagram below (slightly abusing notation)

$$\begin{tikzcd} L \cap V \arrow[r, hook, "\iota"] & M \\ U \times \{0\}. \arrow[u, "\psi^{-1}"] \arrow[ur, "\psi^{-1}"'] & \end{tikzcd}$$

Because the manifold structure on L is given by the subset diffeology inherited from M, all maps in this diagram are smooth. The map on the right, $\psi^{-1} : U \times \{0\} \to M$, is an immersion, and by the chain rule, so is the map on the left, $\psi^{-1} : U \times \{0\} \to L \cap V$. Because both $U \times \{0\}$ and $L \cap V$ are manifolds of dimension $\dim L$, the map on the left, $\psi^{-1} : U \times \{0\} \to L \cap V$, is a diffeomorphism onto its image $L_0 := \psi^{-1}(U \times \{0\})$. Therefore it is a chart of L. In this chart, the inclusion satisfies

$$\begin{tikzcd} L_0 \arrow[r, hook, "\iota"] & V \arrow[d, "\psi"] \\ U \times \{0\} \arrow[u, "\psi^{-1}"] \arrow[r, hook] & U \times W. \end{tikzcd}$$

Therefore ι is locally an inclusion, hence is an immersion. $\square$

The Stefan-Sussmann Theorem

Stefan's definition is ultimately difficult to verify, even when the proposed foliation should be regular. In the regular case, we instead identify foliations with their associated distributions, and apply the Frobenius theorem. Stefan [Ste74] and Sussmann [Sus73] extended the Frobenius theorem simultaneously but independently in 1973-1974 to the singular setting, in what is now called the Stefan-Sussmann theorem. However, while similar, their contributions were not identical, and we will take care here to highlight their separate work.

Stefan worked with singular foliations generated by arrows. These are simply certain 1-plots of $\mathrm{Diff}_{\mathrm{loc}}(M)$, the space of diffeomorphisms between open subsets of M equipped with its functional diffeology (for a description of this diffeology, see [IZ13]).

Definition 3.12. An *arrow* on a manifold M is a 1-plot $a : U \to \mathrm{Diff}_{\mathrm{loc}}(M)$, such that $a(0) = \mathrm{id}$ and $\mathrm{dom}(a(t)) \subseteq \mathrm{dom}(a(s))$, whenever $0 \leq |s| \leq |t|$. We allow for $a(t)$ to be the empty map.

2 recall the topology on L is finer than the subspace topology.

Given a collection $\mathcal{A}$ of arrows, the union of their images, $\bigcup_{a\in\mathcal{A}} a(U)$ generates a pseudogroup (Definition 2.33), which we denote $\Psi\mathcal{A}$. Also associated to $\mathcal{A}$ are two smooth singular distributions, by which we mean:

Definition 3.13. A *(smooth) singular distribution* on a manifold M is a subset $\Delta \subseteq TM$ such that, for every $x \in M$, the set $\Delta_x := \{v \in \Delta \mid \pi(v) = x\}$ is a vector subspace of T_xM, and for every $v \in \Delta$, there is a locally defined vector field X such that $X_{\pi(v)} = v$ and $X_y \in \Delta_y$ for all $y \in \operatorname{dom} X$.

Namely, set

$$\begin{aligned}(\Delta\mathcal{A})_x &:= \left\{ \frac{d}{dt}\Big|_{t=t_0} a(t)(y) \mid a(t)(y) = x \right\} \\ \overline{(\Delta\mathcal{A})}_x &:= \{d\varphi_y v \mid \varphi \in \Psi\mathcal{A} \text{ and } v \in (\Delta\mathcal{A})_x \text{ and } \varphi(y) = x\}.\end{aligned}$$

Stefan [Ste74, Theorem 1] proved:

Theorem 3.14 (Stefan). *Given a collection of arrows $\mathcal{A}$ on M, the partition $\mathcal{F}$ of M into the orbits of $\Psi\mathcal{A}$ is a singular foliation, and $T_xL = \overline{(\Delta\mathcal{A})}_x$ for each $x \in M$, where L is the leaf about x.*

Example 3.15. Consider two vector fields on $\mathbb{R}^2$,

$$X := \frac{\partial}{\partial x} \quad Y := \varphi(x)\frac{\partial}{\partial y},$$

where φ is a smooth, bounded, non-negative, increasing function which vanishes if and only if $x \leq 0$, and $\varphi(x) = 1$ if $x \geq 1$. Let $\mathcal{A}$ consist of the flows of X and Y. Then $\Psi\mathcal{A}$ acts transitively on $\mathbb{R}^2$: it clearly acts transitively on $x > 0$; if $x \leq 0$, first flow along X until we enter the region $x > 0$, flow along $\pm Y$ until we arrive at the required y-coordinate, and then flow along $-X$. It follows that $\overline{(\Delta\mathcal{A})}_{(x,y)} = \mathbb{R}^2$ at all (x, y), and the only orbit of $\Psi\mathcal{A}$ is $L = \mathbb{R}^2$.

In the setting of Stefan's Theorem 3.14, the fact that for each $x \in M$, we have $\overline{(\Delta\mathcal{A})}_x = T_xL$ (for the leaf L about x) means that the singular distribution $\overline{\Delta\mathcal{A}}$ is integrable, in the sense of the next definition below.

Definition 3.16. A singular distribution Δ is *integrable* if, at every $x \in M$, it admits an *integral submanifold*, which is an immersed submanifold (L, σ_L) containing x such that $T_y(L, \sigma_L) = \Delta_y$ for all $y \in L$.

Suppose now that D is a collection of locally defined vector fields on M inducing a singular distribution Δ, and $\mathcal{A}_D$ is the collection of plots of the form $t \mapsto \Phi^X(t, \cdot)$, where Φ^X is the flow of an element of D. We have $\Delta\mathcal{A}_D = \Delta$. Stefan's theorem implies

Corollary 3.17. *The orbits of $\Psi\mathcal{A}_D$ are integral submanifolds of Δ if and only if $\Delta = \overline{\Delta}$.*

This corollary characterizes all integral singular distributions Δ whose integral submanifolds are the orbits of $\Psi\mathcal{A}_D$, for some (hence any) collection of locally defined vector fields D spanning Δ. However, assuming only that Δ is integrable, it is not obvious from the above statements that $\Delta = \overline{\Delta}$. We do not know *a priori* that the integral submanifolds of Δ are the orbits of $\Psi\mathcal{A}_D$. Sussmann [Sus73, Theorem 4.2] showed that nevertheless, for an integrable singular distribution Δ, we do in fact have $\Delta = \overline{\Delta}$. He achieves this as a consequence of his Orbit Theorem [Sus73, Theorem 4.1]), which we state for completeness.

Definition 3.18. An integral submanifold (L, σ_L) of Δ is *maximal* if it is connected, and every connected integral submanifold (S, σ_S) of Δ which intersects L is an open subset of L (in its manifold topology).[3]

Theorem 3.19 (Sussmann's orbit theorem)**.** *Given a singular distribution Δ and set of locally-defined vector fields D as above, the orbits of $\Psi\mathcal{A}_D$ are maximal integral submanifolds of $\overline{\Delta}$.*

Combining Stefan and Sussmann's results yields what is today called the Stefan-Sussmann theorem.

Theorem 3.20 (Stefan-Sussmann)**.** *A singular distribution Δ is integrable if and only if $\Delta = \overline{\Delta}$, in which case its maximal integral submanifolds are leaves of a singular foliation $\mathcal{F}$ such that $T_xL = \Delta_x$, where L is the leaf through x.*

Remark 3.21. Stefan called the condition $\Delta\mathcal{A} = \overline{\Delta\mathcal{A}}$ *homogeneity* of the set of arrows $\mathcal{A}$. Sussmann worked with a set D of partially-defined vector fields spanning a singular distribution Δ, and used the term *D-invariance* for the condition $\Delta = \overline{\Delta}$. We will adopt Stefan's term "homogeniety." This condition is implicit in earlier work by Lobry [Lob70], cf. [Lav18, page 53].

When $\dim \Delta_x$ is constant over M, homogeneity of Δ is equivalent to involutivity, and we recover the global Frobenius theorem 3.1 for regular foliations. In fact, Deahna [Dea40], who provided necessary and sufficient conditions for integrability of a regular distribution before Frobenius, gave homogeniety of Δ, and not involutivity, as the sufficient condition, cf. [Sam01, page 525].

We already remarked that only Sussmann proved maximality of the integral submanifolds of an integrable Δ. Only Stefan proved the orbits of $\Psi\mathcal{A}$ were weakly-embedded submanifolds, and not merely immersed, and only Stefan showed the orbits (i.e. the maximal integral submanifolds) assemble into a singular foliation.

3 Equivalently, (L, σ_L) if, at each point in L, it is maximal if it is maximal with respect to the inclusion.

Example 3.22. Let $A \implies M$ be a Lie algebroid, with anchor $\rho : A \to TM$. The image of the anchor, $\rho(A)$, is an integrable singular distribution. In this case, a maximal integral submanifold of Δ through $x \in M$ coincides with the set of all points reachable from x via A-paths, which are paths γ in M for which there exists a path $\tilde{\gamma}$ in A such that $\rho \circ \tilde{\gamma} = \dot{\gamma}$.

By passing to its associated Lie algebroid, one finds that the connected components of the orbits of a Lie groupoid $G \rightrightarrows M$, and the connected components of the orbits of a Lie group G acting smoothly on M, assemble into a singular foliation.

Historically, integrability of $\rho(A)$ follows from a result of Hermann [Her62] that predates Stefan and Sussmann's work. Hermann showed that any *locally finitely generated Lie sub-algebra*[4] of $\mathfrak{X}(M)$ induces an integrable singular distribution, and we can show that $\rho(A)$ satisfies this hypothesis. That the integral submanifolds are maximal, and assemble into a singular foliation, still requires Stefan's contribution to the Stefan-Sussmann theorem.

As a corollary, we can completely describe singular foliations by their associated singular distributions. This generalizes the Frobenius theorem as stated in the Introduction.

Corollary 3.23. *Given a singular foliation $\mathcal{F}$, the collection $T\mathcal{F} := \bigcup_{x \in M} T_x L$ is an integrable singular distribution. Given an integrable singular distribution Δ, the partition of M into maximal integral submanifolds is a singular foliation, denoted $\mathcal{F}_\Delta$. The assignments*

$$\mathcal{F} \mapsto T\mathcal{F}, \quad \Delta \mapsto \mathcal{F}_\Delta$$

are inverses of each other.

Remark 3.24. We make two comparisons to the regular case. First, homogeneity (see Remark 3.21) implies involutivity, but for non-regular singular distributions, homogeneity is stronger than involutivity. Indeed, the singular distribution spanned by X and Y in Example 3.15 is involutive, but it is not integrable. For regular distributions, involutivity and homogeneity are equivalent.

Second, the maximal integral submanifolds of an integrable regular distribution may also be described as equivalence classes of the relation: $x \sim y$ if there is a path γ from x to y tangent to Δ, i.e. $\dot{\gamma}(t) \in \Delta_{\gamma(t)}$ for all t. In general, this is not the case for non-regular singular distributions. For example, consider the singular distribution spanned by the Euler vector field $X := x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ on $\mathbb{R}^2$. This has maximal integral submanifolds the

4 meaning a Lie subalgebra D of $\mathfrak{X}(M)$ such that, for every open $U \subseteq M$, there exist $X^1, \ldots, X^k \in D$ such that $D|_U \subseteq C^\infty(U)\text{span}(X^1|_U, \ldots, X^k|_U)$.

origin $\{(0,0)\}$, and the rays from the origin. However, every two points can be joined by a path tangent to Δ.

Stefan's results and diffeology

While Stefan's Theorem 3.14 is the most cited part of his paper [Ste74], in the same work he also proved a result about orbits of subgroups of $\mathrm{Diff}(M)$, which has applications to diffeology. In this section, the *D-topology*, and in particular *D-connectedness* of a diffeological space X is central.[5] The D-topology on X is the finest topology for which all the plots are continuous (Definition 2.4), and its D-connected components coincide with its (smooth) path-connected components. See Chapters 2 and 5 of [IZ13] for details.

Stefan proved [Ste74, Theorem 3]:

Corollary 3.25. *Let G be a D-connected diffeological subgroup of* $\mathrm{Diff}(M)$*. Then the G-orbits assemble into a singular foliation of M.*

Proof. Let $\mathcal{A}$ consist of all plots (paths) $a : \mathbb{R} \to G$ with $a(0) = \mathrm{id}$. Then $\mathcal{A}$ is a set of arrows. Because G is D-connected, it is path-connected ([IZ13, Article 5.7]). Thus $\bigcup_{a \in \mathcal{A}} a(\mathbb{R}) = G$, and $\Psi\mathcal{A}$ is the pseudogroup generated by G, so the result follows from Theorem 3.14. □

Conversely, we have:

Proposition 3.26. *Every singular foliation of M is a partition of M into orbits of some D-connected subgroup of* $\mathrm{Diff}(M)$*.*

Proof. Fix a singular foliation $\mathcal{F}$. For each $x \in M$, take a collection $\{X^x\}$ of compactly supported vector fields whose values at x span $T_x\mathcal{F}$. This is possible because $T\mathcal{F}$ is smooth (Definition 3.13, Corollary 3.23), and M supports bump functions. Let $\mathcal{A}$ denote the collection of all plots of the form $t \mapsto \Phi^{X^x}(t, \cdot)$. Then $\bigcup_{a \in \mathcal{A}} a(\mathbb{R})$ generates some D-connected subgroup of $\mathrm{Diff}(M)$, and its orbits coincide with the orbits of $\Psi\mathcal{A}$, which are precisely the leaves of $\mathcal{F}$ by the Stefan-Sussmann Theorem 3.20. □

Remark 3.27. In the proof above, we required a collection of vector fields which spanned $T\mathcal{F}$. We were not concerned with how many vector fields were necessary. In 2008 and 2012, respectively, Sussmann [Sus08] and Drager, Lee, Park, and Richardson [DLPR12] independently showed that only finitely many are required. Precisely, they showed that given a singular distribution Δ, there exist finitely many vector fields $X^1, \ldots, X^n$ such

5 Here the "D" stands for "Diffeology." In this context, it does not represent some family of vector fields.

that $\Delta_x = \mathrm{span}\{X^1_x, \ldots, X^n_x\}$ for every $x \in M$. Drager et al. provided some bounds for n. Note, however, that the collection $X^1, \ldots, X^n$ does not necessarily generate the sheaf of sections of Δ. In fact, Drager et. al. [DLPR12, Section 5] give an example of a singular distribution over $\mathbb{R}$ whose sheaf of sections is not finitely generated.

As an immediate of Proposition 3.26, we have:

Corollary 3.28. *For a singular foliation $\mathcal{F}$, the group of diffeomorphisms of M which fix the leaves of $\mathcal{F}$ acts transitively on each leaf.*

It follows that, for a manifold M, it is equivalent to consider singular foliations, to consider integrable distributions, and to consider orbits of D-connected subgroups of $\mathrm{Diff}(M)$. Here is one consequence of this fact.

Proposition 3.29. *The partition given by a singular foliation $\mathcal{F}$ of M satisfies the frontier condition: if S and L are leaves, and $S \cap \overline{L} \neq \emptyset$, then $S \subseteq \overline{L}$ (here the closure is taken relative to M).*

Proof. Fix a D-connected subgroup G of $\mathrm{Diff}(M)$ whose orbits are the leaves of $\mathcal{F}$. It suffices to show $\overline{L}$ is G-invariant. Suppose $x \in \overline{L}$, and let (x_n) be a sequence of points in L which converge (in M) to x. Then for any $g \in G$, we have $g(x_n) \to g(x)$, and all the $g(x_n)$ are in L, hence $g(x) \in \overline{L}$. Since g was arbitrary, this completes the proof. $\square$

Androulidakis-Skandalis singular foliations

We view singular foliations as partitions of a manifold into leaves, and have seen that by the Stefan-Sussmann theorem, this is equivalent to considering integrable singular distributions. Some authors take a different approach, and view singular foliations as choices of submodules of vector fields. For instance, here is the definition given by Androulidakis and Skandalis [AS09].

Definition 3.30. An *Androulidakis-Skandalis singular foliation* is a $C^\infty(M)$-submodule $\mathscr{D}$ of $\mathfrak{X}_c(M)$ that is locally finitely generated[6] and involutive.

We call these submodules "Androulidakis-Skandalis singular foliations" to distinguish them from the singular foliations of Definition 3.6, which are partitions of manifolds into leaves. In the literature, such submodules are often called simply "singular foliations." The Stefan-Sussmann Theorem 3.20 implies that the singular distribution associated to an

6 this means that, about every $x \in M$, there is a neighbourhood U and $Y^1, \ldots, Y^k \in \mathscr{D}|_U$ such that $\mathscr{D}|_U = C^\infty_c(U)\mathrm{span}(Y^1, \ldots, Y^k)$, where $\mathscr{D}|_U$ is the $C^\infty(U)$-submodule of $\mathfrak{X}_c(U)$ generated by all the $Y = fX|_U$, where $f \in C^\infty_c(U)$ and $X \in \mathscr{D}$.

Androulidakis-Skandalis singular foliation is integrable, and accordingly some authors, e.g. [AZ16], have called Androulidakis-Skandalis singular foliations "Stefan-Sussmann singular foliations."

It is not necessary, however, to invoke the Stefan-Sussmann theorem to deduce integrability of Androulidakis-Skandalis singular foliations: in this case Hermann's theorem [Her62] (stated in Example 3.22), which predates Stefan and Sussmann's work, implies the same result. Moreover, Hermann himself used "foliation with singularities" to refer not to a partition of a manifold into leaves, but to a locally finitely generated involutive Lie subalgebra of $\mathfrak{X}(M)$. This positions Hermann as an important, early proponent of the opinion that a singular foliation of a manifold is a finer notion than a partition into leaves.

Remark 3.31. As noted, for example, by Garmendia and Zambon [GZ19, Remark 1.8], and by Wang [Wan17, Remark 2.1.13], it is equivalent to define an Androulidakis-Skandalis singular foliation as an involutive, locally finitely generated subsheaf of the sheaf of vector fields on M. This approach could allow a definition of singular foliations on possibly non-Hausdorff manifolds, where the submodule and sheaf theoretic notions no longer coincide, cf. Remark 3.46.

Many different choices of Androulidakis-Skandalis singular foliation may induce the same singular foliation, as the example below shows.

Example 3.32. Let $\mathscr{D}_k$ be generated by $x^k \frac{\partial}{\partial x}$. Each $\mathscr{D}_k$ induces the Stefan singular foliation of $\mathbb{R}$ with three leaves: $x < 0$, $\{0\}$, and $x > 0$. However, no two of the modules $\mathscr{D}_k$ are isomorphic.

Every Lie algebroid, hence every Lie groupoid and action of a Lie group on a manifold, induces an Androulidakis-Skandalis singular foliation. On the other hand, Androulidakis and Zambon [AZ13, Proposition 1.3] have described an Androulidakis-Skandalis singular foliation $\mathscr{D}_{\text{counter}}$ of $\mathbb{R}^2$ that is not induced by any Lie algebroid. We do not give the details here, but note that the underlying Stefan singular foliation is the partition of $\mathbb{R}^2$ into point leaves $\{(k, 0)\}$ for natural k, and the complement $\mathbb{R}^2 \setminus \bigcup_{k \geq 1}\{(k, 0)\}$. This underlying Stefan singular foliation is induced by a smooth Lie group action on $\mathbb{R}^2$, namely that generated by vector fields $f(x, y)\frac{\partial}{\partial x}$ and $f(x, y)\frac{\partial}{\partial y}$, where f is a bounded, non-negative function that vanishes precisely on the point leaves $\{(k, 0)\}$. But this Lie group action does not induce the Androulidakis-Skandalis singular foliation $\mathscr{D}_{\text{counter}}$.

At time of writing, the following question is open:

Question 3.33. Is every Stefan singular foliation induced by some Androulidakis-Skandalis singular foliation?

As we see in Example 3.32 above, even if the answer is affirmative, the choice of Androulidakis-Skandalis singular foliation is not unique. It is also not natural, in general: consider the singular foliation $\mathcal{F}$ of $\mathbb{R}$ whose leaves are $\{x\}$ with $x \leq 0$, and $x > 0$. By [DLPR12, Proposition 5.3], the space of all vector fields tangent to $\mathcal{F}$ is not locally finitely generated, and it is not clear how to choose a "best" vector field tangent to $\mathcal{F}$ to use as a generator of an Androulidakis-Skandalis singular foliation.

3.2 TRANSVERSE EQUIVALENCE OF SINGULAR FOLIATIONS

Given a singular foliation $\mathcal{F}$ of M, the leaf space $M/\mathcal{F}$ is naturally a diffeological space. For convenience, we recall the plots of $M/\mathcal{F}$.

Definition 3.34. A map $p : \mathsf{U} \to M/\mathcal{F}$, where U is an open subset of Cartesian space, is a *plot* of the quotient diffeology of $M/\mathcal{F}$ if, about each $r \in \mathsf{U}$, there is an open neighbourhood V or r, and a smooth map $q : \mathsf{V} \to M$, such that $p|_{\mathsf{V}} = \pi \circ q$, where π is the quotient map.

The D-topology coincides with the quotient topology [IZ13, Article 2.12]. However, whereas the quotient topology on $M/\mathcal{F}$ may be trivial, or lose information about the singular foliation $\mathcal{F}$, its diffeology is often richer, as the following examples show.

Example 3.35. Let $\pi : \mathbb{R}^2 \to T^2 := \mathbb{R}^2/\mathbb{Z}^2$ denote the quotient map for the $\mathbb{Z}^2$ action on $\mathbb{R}^2$. For irrational α, let S_α be the line in $\mathbb{R}^2$ with slope α. The conjugacy classes of the subgroup $\pi(S_\alpha)$ of T^2 assemble into a regular foliation of T^2. We call the leaf space $T_\alpha := T^2/\pi(S_\alpha)$ an *irrational torus.*

The quotient topology on T_α is always trivial, but the quotient diffeology is not: Donato and Iglesias-Zemmour [DI85] proved that $T_\alpha \cong T_\beta$ if and only if $\beta = \frac{a+b\alpha}{c+d\alpha}$, where $a, b, c, d \in \mathbb{Z}$ and $ad - bc = \pm 1$.

Example 3.36. Consider the action of the orthogonal group $O(n)$ on $\mathbb{R}^n$. The leaves of the induced singular foliation are the orbits of the action, namely the origin $\{0\}$ and the concentric spheres. Topologically, the quotient spaces $\mathbb{R}^n/O(n)$ are all homeomorphic to the half-line $[0, \infty)$ with its subspace topology. But by [IZ13, Exercise 50,51] they are all diffeologically distinct, and none are diffeologically diffeomorphic to $[0, \infty)$ with its subset diffeology. For $n = 1$, we get a diffeological orbifold $\mathbb{R}/\mathbb{Z}_2$.

We may, therefore, propose the diffeological space $M/\mathcal{F}$ as a model for the transverse geometry of $\mathcal{F}$. In the next sections, we compare this notion with another model.

Molino Transverse Equivalence

In [Mol88, Definition 2.1], Molino defines a notion of transverse equivalence for regular foliations. In [GZ19], Garmendia and Zambon extended this notion to Androulidakis-Skandalis singular foliations, and our definition below is similar to theirs and extends Molino's. However, due to the differences between Androulidakis-Skandalis and Stefan singular foliations outlined in Section 3.1, our definition does not coincide with Garmendia and Zambon's.

We need two technical Lemmas first.

Lemma 3.37. *Suppose $p : M \to N$ is a surjective submersion with connected fibers, and assume N is connected. Then M is connected.*

Proof. Take any continuous function $g : M \to \{0, 1\}$. Because the fibers of p are connected, g is constant on the fibers. Therefore, as p is a submersion, there is a smooth function $h : N \to \{0, 1\}$ such that $h \circ p = g$. Since N is connected, $h(N)$ is a single point; since p is surjective, $g(M) = h(N)$ is a single point. Therefore every continuous function $g : M \to \{0, 1\}$ is constant, and we conclude that M is connected. □

Lemma 3.38. *Suppose $p : M \to N$ is a submersion. For every locally defined vector field Y on N, and every $v \in T_xM$ such that $dp_x(v) = Y_{p(x)}$, there is a locally defined vector field X on M such that $X_x = v$ and X is p-related to Y.*

Proof. Because p is a submersion, without loss of generality we may assume that $p : \mathbb{R}^m \to \mathbb{R}^n$ is the projection of the first n coordinates. For $v = (v^1, \dots, v^m) \in T_xM$, define

$$X_{x'} := (Y_{p(x')}, v^{n+1}, \dots, v^m).$$

Then X is p-related to Y, because $dp_{x'}$ is the projection of the first n coordinates, and the condition $dp_x(v) = Y_{p(x)}$ writes

$$(v^1, \dots, v^n) = Y_{p(x)},$$

so $X_x = v$. □

Proposition 3.39. *Let $p : M \to N$ be a surjective submersion with connected fibers. If Δ is an integrable singular distribution on N, then $(dp)^{-1}(\Delta)$ is an integrable singular distribution on M. If L is a leaf of the singular foliation of N induced by Δ, then $p^{-1}(L)$ is a leaf of the singular foliation of M induced by $(dp)^{-1}(\Delta)$.*

Proof. To see that $(dp)^{-1}(\Delta)$ is a smooth singular distribution, take $v \in ((dp)^{-1}(\Delta))_x$. Then $dp_x(v) \in \Delta_{p(x)}$, and since Δ is smooth, there is some partially defined vector field Y of N tangent to Δ with $dp_x(v) = Y_{p(x)}$. By Lemma 3.38, we get a partially defined vector field X of M such that $X_x = v$, and $dp(X) = Y$. This vector field is tangent to $(dp)^{-1}(\Delta)$, and passes through v. Hence $(dp)^{-1}(\Delta)$ is smooth.

To see that $(dp)^{-1}(\Delta)$ is integrable, let $x \in M$ and let L be the maximal integral submanifold of Δ through $p(x)$. Because p is a surjective submersion, $p^{-1}(L)$ is a weakly-embedded submanifold of M, and for each $x' \in p^{-1}(L)$,

$$T_{x'}(p^{-1}(L)) = (dp_{x'})^{-1}(T_{p(x')}L) = ((dp)^{-1}(\Delta))_{x'}.$$

The Stefan-Sussmann Theorem 3.20 implies that the maximal integral submanifolds of $(dp)^{-1}(\Delta)$ assemble into a singular foliation of M. But the $p^{-1}(L)$ already partition M into connected (by Lemma 3.37) integral submanifolds of $(dp)^{-1}(\Delta)$. Corollary 3.23 lets us conclude that the $p^{-1}(L)$ are the maximal integral submanifolds of $(dp)^{-1}(\Delta)$. □

In light of Proposition 3.39, we can use the following notation.

Definition 3.40. Given a surjective submersion with connected fibers $p : M \to N$, and an integrable singular distribution Δ on N with associated singular foliation $\mathcal{F}$, let $p^{-1}\Delta := (dp)^{-1}(\Delta)$, and let $p^{-1}\mathcal{F}$ denote the singular foliation associated to $p^{-1}\Delta$; its leaves are the sets $p^{-1}(L)$, where the L are leaves of $\mathcal{F}$. We call $p^{-1}\Delta$ and $p^{-1}\mathcal{F}$ the *pullbacks* of Δ and $\mathcal{F}$.

Using the pullback foliation, we now define Molino transverse equivalence of singular foliations.

Definition 3.41. Two singular foliations, $(N_0, \mathcal{F}_0)$ and $(N_1, \mathcal{F}_1)$, are *Molino transverse equivalent* if there exists a singular foliation $(M, \mathcal{F})$ and surjective submersions with connected fibers $p_i : M \to N_i$ such that $p_i^{-1}(\mathcal{F}_i) = \mathcal{F}$. We will write $\mathcal{F}_0 \cong \mathcal{F}_1$.

Naturally, we must show:

Proposition 3.42. *Molino transverse equivalence is an equivalence relation on singular foliations.*

Proof. Reflexivity is witnessed by the identity. Symmetry is clear because we can reverse the roles of the N_i. For transitivity, build the following diagram:

$$\begin{tikzcd}
 & & M \,{}_{p_1}\!\times_{p_1'} M' \arrow[dl, "\mathrm{pr}_1"'] \arrow[dr, "\mathrm{pr}_2"] & & \\
 & (M, \mathcal{F}) \arrow[dl, "p_0"'] \arrow[dr, "p_1"] & & (M', \mathcal{F}') \arrow[dl, "p_1'"'] \arrow[dr, "p_2'"] & \\
(N_0, \mathcal{F}_0) & & (N_1, \mathcal{F}_1) & & (N_2, \mathcal{F}_2).
\end{tikzcd}$$

The bottom two rows indicate the assumed Molino transverse equivalences. Denote the fiber product at the top by M''. This is a manifold because p_1 is a submersion. The projections are surjective submersions because the p_i and p_i' are surjective submersions, thus so too are the compositions $p_0 \circ \mathrm{pr}_1$ and $p_2' \circ \mathrm{pr}_2$. We claim that, moreover, these have connected fibers, and pull back $\mathcal{F}_0$ and $\mathcal{F}_2$, respectively, to the same singular foliation of M''.

First, in light of Lemma 3.37, it suffices to show, without loss of generality, that pr_1 has connected fibers. Let $x \in M$, and take (x,z) and (x,z') in M''. Then z and z' are in the same fiber of p_1', which is connected, hence there is a path γ' joining them in this fiber. Then $(x, \gamma'(t))$ is a path in the fiber of pr_1 joining (x,z) and (x,z'), as required.

The second assertion follows from the fact that any leaf L of $\mathcal{F}$ has the form $p_0^{-1}(L_0) = p_1^{-1}(L_1)$, and any leaf L' of $\mathcal{F}'$ has the form $(p_1')^{-1}(L_1') = (p_2')^{-1}(L_2')$, and from using the usual property of pullbacks that $(p_0 \circ \mathrm{pr}_1)^{-1}\mathcal{F}_0 = \mathrm{pr}_1^{-1} p_0^{-1}\mathcal{F}_0$, etc. This proves that $(M'', (p_0 \circ \mathrm{pr}_1)^{-1}(\mathcal{F}_0))$, with the surjective submersions $p_0 \circ \mathrm{pr}_1$ and $p_2' \circ \mathrm{pr}_2$, gives a Molino transverse equivalence between $\mathcal{F}_0$ and $\mathcal{F}_2$. $\square$

We now relate Molino transverse equivalence and diffeology. One direction is straightforward: a Molino transverse equivalence induces a diffeological diffeomorphism of the leaf spaces.

Proposition 3.43. *Suppose $p : M \to N$ is a surjective submersion with connected fibers, and let $\mathcal{F}$ be a singular foliation of N. Then the map*

$$\varphi : N/\mathcal{F} \to M/p^{-1}\mathcal{F}, \quad L \mapsto p^{-1}(L)$$

is a diffeological diffeomorphism. Consequently, if $p_i : M \to N_i$ witness a Molino transverse equivalence between $\mathcal{F}_i$, then the map

$$N_0/\mathcal{F}_0 \to N_1/\mathcal{F}_1, \quad L \mapsto p_1(p_0^{-1}(L))$$

is a diffeological diffeomorphism.

Proof. The map φ is well-defined by definition of $p^{-1}\mathcal{F}$ (here we use the fact that p has connected fibers, as in Proposition 3.39), and its inverse is $p^{-1}(L) \mapsto L$. Both φ and φ^{-1} fit in the following diagram:

$$\begin{tikzcd}
M \arrow[r, "p"] \arrow[d, "\pi_1"] & N \arrow[d, "\pi_2"] \\
M/p^{-1}(\mathcal{F}) \arrow[r, bend left, "\varphi^{-1}"] & N/\mathcal{F}. \arrow[l, bend left, "\varphi"]
\end{tikzcd}$$

Because π_1 and $\pi_2 \circ p$ are subductions, by Lemma 2.9 so are φ and φ^{-1}. □

Example 3.44. Consider the singular foliations induced by the action of $O(n)$ on $\mathbb{R}^n$, from Example 3.36. Because the $\mathbb{R}^n/O(n)$ are not diffeomorphic for different n, the singular foliations of $\mathbb{R}^n$ are not Molino transversely equivalent.

The converse is more subtle. Indeed, there exist two regular foliations, $(N_i, \mathcal{F}_i)$, with diffeomorphic leaf spaces, but which are not Molino transverse equivalent. We will see this by detouring through Morita equivalence of Lie groupoids.

Molino versus Morita equivalence

We saw in Example 3.22 that every Lie groupoid induces a singular foliation on its base manifold. For Lie groupoids, there is a long established notion of transverse, or weak, equivalence, called *Morita equivalence*. In this subsection, we compare the notions of Molino and Morita equivalence.

Proposition 3.45. *Let $G \rightrightarrows M$ and $H \rightrightarrows N$ be two source-connected Lie groupoids with Hausdorff arrow spaces. If G and H are Morita equivalent, then the singular foliations $\mathcal{F}_G$ and $\mathcal{F}_H$ on M and N are Molino transverse equivalent.*

Proof. Fix an invertible bibundle $P : G \to H$ (see Definition 2.22 for details)

$$\begin{tikzcd} G \circlearrowright & P \arrow[dl, "a"'] \arrow[dr, "a'"] & \circlearrowleft H \\ M & & N. \end{tikzcd}$$

We claim that this exhibits a Molino transverse equivalence. First, since G and H are Hausdorff, by [GZ19, Corollary A.7], we may choose P to be Hausdorff. Both a and a' are surjective submersions by assumption. Since G acts freely and transitively on the fibers of a', the fiber of a' over x_0' is diffeomorphic to $s^{-1}(x_0') \cdot p_0$ for any fixed p_0 in the fiber, and this is connected because G is source-connected. Similarly, the fibers of a are connected.

All that remains is to show the leaves of $a^{-1}\mathcal{F}_G$ coincide with the leaves of $(a')^{-1}(\mathcal{F}_H)$. Let $\mathcal{O}$ be the orbit through a fixed x_0 in M. Choose $p_0 \in P$ with $a(p_0) = x_0$, and let $\mathcal{O}'$ be the orbit of $a'(p_0)$ in N. We claim $a^{-1}(\mathcal{O}) = (a')^{-1}(\mathcal{O}')$.

Letting $a(p) \in \mathcal{O}$, we may show $a'(p) \in \mathcal{O}'$; the converse direction is similar. There is an arrow $g : x_0 \to a(p)$. By G-invariance of a', both $g \cdot p_0$ and p are in the same fiber of a'. Since the H action is transitive on this fiber, there is some arrow h with $g \cdot p_0 \cdot h = p$. This action is only possible if h is an arrow $h : a'(p) \to a'(g \cdot p_0) = a'(p_0)$. Thus $a'(p)$ and $a'(p_0)$ are in the same orbit $\mathcal{O}'$, as required. □

Remark 3.46. The Hausdorff assumption is material, because if G and H are not Hausdorff, it is not clear that we may choose the invertible bibundle P above to be Hausdorff. One possible attempt to treat the case of non-Hausdorff Lie groupoids is to modify the definition of Molino transverse equivalence to allow for non-Hausdorff manifolds. But doing so begs the question of how to define singular foliations on non-Hausdorff manifolds, which we leave for another time. Garmendia and Zambon encounter a similar problem in the case of Androulidakis-Skandalis singular foliations, and suggest it may be remedied by viewing Androulidakis-Skandalis singular foliations as sheaves instead of submodules, as in Remark 3.31; see [GZ19, Section 4.1].

Given an arbitrary singular foliation $(N, \mathcal{F})$, it is unknown whether there is a Lie groupoid $G \rightrightarrows N$ which induces $\mathcal{F}$. Furthermore, it is possible that two non-Morita equivalent Lie groupoids induce the same singular foliation.

Example 3.47. Consider the actions of the general linear group $\mathrm{GL}(n)$, and the special linear group $\mathrm{SL}(n)$, on $\mathbb{R}^n$. Both induce the same singular foliation of $\mathbb{R}^n$, whose leaves are the origin and its complement. However, the action groupoids $\mathrm{GL}(n) \ltimes \mathbb{R}^n$ and $\mathrm{SL}(n) \ltimes \mathbb{R}^n$ are not Morita equivalent. This is because their stabilizer groups at the origin are not isomorphic.

For regular foliations $(N, \mathcal{F})$, however, we have a distinguished groupoid inducing the foliation, the holonomy groupoid $\mathrm{Hol}(\mathcal{F})$ (see Chapter 4, Section 2.3 for its definition and details). It is reasonable to ask whether Molino transverse equivalence of regular foliations implies their holonomy groupoids are Morita equivalent, and indeed this is the case:

Proposition 3.48. *If two regular foliations are Molino transverse equivalent, then their holonomy groupoids are Morita equivalent.*

This coincides with Proposition 3.30 in Garmendia and Zambon's paper [GZ19], whose proof in the regular case is outlined after their Theorem 3.21. We recall that Garmendia and Zambon work with Androulidakis-Skandalis singular foliations, and not Stefan singular foliations. However, in the regular case, these notions coincide, so their proof of this statement works in this setting and we refer the reader there for details.

Remark 3.49. In fact, Garmendia and Zambon prove a stronger statement. As we discussed in Section 3.1, Androulidakis-Skandalis singular foliations are not generally induced by any Lie groupoid. However, Androulidakis and Skandalis in [AS09] constructed an open topological groupoid for every Androulidakis-Skandalis singular foliation $\mathcal{F}$, called the *holonomy groupoid* of $\mathcal{F}$, which coincides with the usual holonomy groupoid in the regular case. Garmendia and Zambon proved that, provided the holonomy groupoids are Hausdorff, they are Morita equivalent (as open topological groupoids) if and only if the associated singular foliations are what they call Hausdorff Morita equivalent.

Corollary 3.50. *Two regular foliations with Hausdorff holonomy groupoids are Molino transverse equivalent if and only if their holonomy groupoids are Morita equivalent.*

We are now prepared to illustrate the example alluded to at the end of Section 3.2, of two regular foliations with diffeomorphic leaf spaces, but which are not Molino transverse equivalent. This example also appears in more detail in [KM22], and in Chapter 5 Section 5.4.

Example 3.51. Let $h : \mathbb{R} \to \mathbb{R}$ be a smooth non-negative function that is flat[7] at 0 and is positive everywhere else, such that the vector field $X := h\frac{\partial}{\partial x}$ is complete. Let $\psi := \Phi_1^X$ denote the time-1 flow of X, and set

$$\hat{\psi}(x) := \begin{cases} \psi(x) & \text{if } x \geq 0 \\ \psi^{-1}(x) & \text{if } x < 0. \end{cases}$$

Both ψ and $\hat{\psi}$ are smooth. By iterating ψ, we get a $\mathbb{Z}$-action on $\mathbb{R}$, and also one on $\mathbb{R}^2$ given by

$$k \cdot (t, x) := (t + k, \psi^k(x)).$$

This action preserves the foliation of $\mathbb{R}^2$ by horizontal lines (but not the leaves themselves), and the action is free and properly discontinuous. Therefore, passing to the quotient, $M_\psi := \mathbb{R}^2/\psi$ is a manifold and we have the quotient foliation $\mathcal{F}_\psi$. Moreover, $M_\psi/\mathcal{F}_\psi$ is diffeomorphic to $\mathbb{R}/\psi$. Similarly, we can form $M_{\hat{\psi}}/\mathcal{F}_{\hat{\psi}}$, and this is diffeomorphic to $\mathbb{R}/\hat{\psi}$. But $\mathbb{R}/\psi = \mathbb{R}/\hat{\psi}$, so we have two foliations foliation with diffeologically diffeomorphic leaf spaces. However, by [KM22, Proposition 7.1], or Section 5.4 of this thesis, the étale holonomy groupoids associated to these foliations are not Morita equivalent. Therefore their holonomy groupoids are not Morita equivalent, hence by Proposition 3.48, the foliations are not transverse equivalent.

Diffeology and Molino equivalence for quasifolds

While in general, a diffeological diffeomorphism between the leaf spaces of singular foliations does not induce a Molino transverse equivalence

7 this means that h and all its derivatives vanish at the point

(cf. Example 3.51), there is a class of regular foliations for which this holds. These are regular foliations whose holonomy groupoids are Morita equivalent to quasifold groupoids, introduced in [KM22].

Definition 3.52. A *n-quasifold groupoid* is a Lie groupoid $G \rightrightarrows M$, with Hausdorff arrow space, such that: for each $x \in M$, there is an open neighbourhood U of x, a countable group Γ acting affinely on $\mathbb{R}^n$, an open subset V of $\mathbb{R}^n$, and an isomorphism[8] of Lie groupoids $G|_U \to (\Gamma \ltimes \mathbb{R}^n)|_V$.

Proposition 3.53. *Assume $(N_i, \mathcal{F}_i)$ are regular foliations, and their holonomy groupoids $\mathrm{Hol}(\mathcal{F}_i)$ are each Morita equivalent to effective quasifold groupoids $G_i \rightrightarrows M_i$. If the leaf spaces $N_i/\mathcal{F}_i$ are diffeologically diffeomorphic, then the foliations $\mathcal{F}_i$ are Molino transverse equivalent.*

Proof. Morita equivalent Lie groupoids have diffeomorphic orbit spaces, therefore the M_i/G_i are diffeomorphic to $N_i/\mathcal{F}_i$, for each $i = 0, 1$. It follows from the assumption that the M_i/G_i are diffeomorphic to each other. Then, because the G_i are effective quasifold groupoids, we use [KM22, Proposition 5.4] to conclude that the G_i are Morita equivalent. Therefore the $\mathrm{Hol}(\mathcal{F}_i)$ are Morita equivalent.

If we show each $\mathrm{Hol}(\mathcal{F}_i)$ is Hausdorff, we may conclude the proof with Corollary 3.50. But to be Hausdorff is invariant under Morita equivalence ([MM03, Proposition 5.3], and the G_i are Hausdorff by definition of quasifold groupoid, hence the holonomy groupoids are Hausdorff, as required. □

Remark 3.54. The assumption that the G_i are effective is unnecessary. Each $\mathrm{Hol}(\mathcal{F}_i)$ is Morita equivalent to its étale holonomy groupoid, and these are effective. To be effective is stable under Morita equivalence ([MM03, Example 5.21 (2)], hence the G_i are necessarily effective.

In Proposition 3.53, we impose a condition on the holonomy groupoid $\mathrm{Hol}(\mathcal{F}_i)$, and not on the foliation itself. There are cases, however, where we can deduce the holonomy groupoid is Morita equivalent to a quasifold groupoid directly from properties of the foliation. We end this brief section with an example.

Example 3.55. A distinguished class of quasifold groupoids are proper étale groupoids, which are sometimes called orbifold groupoids. Their local linear models arise as a consequence of the linearization theorem for proper groupoids, for example see [CS13]; the resulting groups Γ are the stabilizer subgroups G_x, which are finite. If all of the leaves of a regular foliation $\mathcal{F}$ are compact and have finite holonomy groups, by the Reeb stability theorem the holonomy groupoid $\mathrm{Hol}(\mathcal{F})$ is proper

8 invertible functor with smooth inverse

([MM03, Example 5.28 (2)]). Because to be proper is stable under Morita equivalence ([MM03, Proposition 5.26]), we may apply Proposition 3.53 to such foliations $\mathcal{F}$.

3.3 BASIC COHOMOLOGY OF SINGULAR FOLIATIONS

Given a singular foliation $(M, \mathcal{F})$, we have the associated complex of basic differential forms. In this section, we will show that Molino transverse equivalent singular foliations have identical complexes of basic forms, strengthening the notion that the basic complex captures the transverse geometry. We will end with a discussion of the comparison between the complex of basic differential forms, and the complex of diffeological differential forms on the leaf space.

Definition 3.56. A differential form $\alpha \in \Omega^\bullet(M)$ is $\mathcal{F}$-*basic* if, for every vector field X tangent to $\mathcal{F}$, both

$$\iota_X\alpha = 0 \text{ and } \mathcal{L}_X\alpha = 0.$$

The first condition says α is *horizontal*. The second says α is *invariant*. We denote the collection of basic differential forms by $\Omega^\bullet_b(M, \mathcal{F})$. This is a de-Rham subcomplex of $\Omega^\bullet(M)$, with the usual differential. Its cohomology $H^\bullet_b(M, \mathcal{F})$ is the *basic cohomology* associated to $\mathcal{F}$.

Example 3.57. If G is a connected Lie group acting smoothly on M, and $\mathcal{F}$ is the associated singular foliation, the $\mathcal{F}$-basic forms are precisely those that are horizontal and G-invariant. More generally, if G is a source-connected Lie groupoid, and $\mathcal{F}$ is the associated singular foliation, the $\mathcal{F}$-basic forms are those for which $s^*\alpha = t^*\alpha$, see [Miy23a, Proposition 5.5].

Proposition 3.58. *Suppose $p : M \to N$ is a surjective submersion with connected fibers, and that $(N, \mathcal{F})$ is a singular foliation. Then p^* is an isomorphism from $\Omega^\bullet_b(N, \mathcal{F})$ to $\Omega^\bullet_b(M, p^{-1}\mathcal{F})$.*

Proof. Because p is a submersion, p^* is injective. It remains to show p^* is into, and onto. First, let α be an $\mathcal{F}$-basic form on N, and take X tangent to $p^{-1}\mathcal{F}$. Then

$$(\iota_X p^*\alpha)_x = \alpha_{p(x)}(p_*X_x, \cdot) = (\iota_{p_*X_x}\alpha)_x = 0,$$

since p_*X_x is tangent to $\mathcal{F}$. As $d\alpha$ is also $\mathcal{F}$-basic, by a similar computation we have $\iota_X(p^*d\alpha) = 0$, so by Cartan's formula,

$$\mathcal{L}_X p^*\alpha = \iota_X(dp^*\alpha) = \iota_X(p^*d\alpha) = 0.$$

Second, let β be a $p^{-1}\mathcal{F}$-basic form on M. Since p is a surjective submersion with connected fibers, its fibers form a regular foliation $\mathcal{F}'$ of M. Any vector field tangent to these fibers is also tangent to $p^{-1}\mathcal{F}$, so the $p^{-1}\mathcal{F}$-basic forms are all $\mathcal{F}'$-basic. In particular, β is $\mathcal{F}'$-basic. Now, we can identify $N \cong M/\mathcal{F}'$, and we can identify the quotient with $p : M \to N$. By [Lee13, Exercise 14.9], the pullback p^* is an isomorphism from $\Omega^\bullet(N)$ to $\Omega_b^\bullet(M, \mathcal{F}')$. This provides $\alpha \in \Omega^\bullet(N)$ such that $p^*\alpha = \beta$.

We now check that α is $\mathcal{F}$-basic. Take Y tangent to $\mathcal{F}$, and fix $x \in M$. By Lemma 3.38, lift Y to a p-related vector field X about x, which is tangent to $p^{-1}\mathcal{F}$. Since β is horizontal,

$$0 = (p^*\alpha)_x(X, \cdot) = \alpha_{p(x)}(Y_{p(x)}, p_* \cdot).$$

But p and p_* are onto, so we conclude $\iota_Y\alpha = 0$, and α is horizontal. We also check that since β is invariant,

$$p^*(\mathcal{L}_Y\alpha) = L_X p^*\alpha = 0.$$

But p^* is injective, so $\mathcal{L}_Y\alpha = 0$, and α is invariant. This completes the proof. □

Corollary 3.59. *If two singular foliations are Molino transverse equivalent, then their complexes of basic forms, hence their basic cohomologies, are isomorphic.*

Diffeology provides an alternative approach to using differential forms to study the transverse geometry of a singular foliation. Every diffeological space X comes with a de Rham complex of diffeological differential forms, $\Omega^\bullet(X)$. When X is a leaf space of a singular foliation $(N, \mathcal{F})$, we can directly compare $\Omega^\bullet(N/\mathcal{F})$ and $\Omega_b^\bullet(N, \mathcal{F})$. The following discussion is pursued in more detail in Chapter 4, but we give a preliminary treatment here.

Theorem 3.60. *Fix a singular foliation $(N, \mathcal{F})$. The quotient map $\pi : N \to N/\mathcal{F}$ induces a map of chain complexes $\pi^* : \Omega^\bullet(N/\mathcal{F}) \to \Omega^\bullet(N)$. This is injective, and its image is contained in the space of $\mathcal{F}$-basic forms. The pullback π^* is an isomorphism $\Omega^\bullet(N/\mathcal{F}) \to \Omega_b^\bullet(N, \mathcal{F})$ whenever*

- *the set of points in leaves of dimension k is a diffeological submanifold of N, for each k, or*
- *π^* is an isomorphism when we replace $(N, \mathcal{F})$ with $N_{>0}$ (N with the zero-leaves excised) and the induced singular foliation $\mathcal{F}_{>0}$.*

This is the main result of [Miy23a], or of Chapter 4. It implies the earlier results from Karshon and Watts [KW16] and Watts [Wat22], in the case of connected groups and source-connected groupoids, respectively. For regular foliations, this was proved by Hector, Marcías-Virgós, and Sanmartín-Carbón in [HMVSC11]. The following question remains open:

Question 3.61. Is $\pi^* : \Omega^\bullet(N/\mathcal{F}) \to \Omega_b^\bullet(N, \mathcal{F})$ always an isomorphism?

However, Corollary 3.59 lets us show π^* is an isomorphism for an *a priori* larger class of singular foliations than Theorem 3.60.

Proposition 3.62. *Suppose $(N_i, \mathcal{F}_i)$ are Molino transverse equivalent singular foliations, and assume that $\pi_i : N_i \to N_i/\mathcal{F}_i$ induces an isomorphism $\pi_i^* : \Omega^\bullet(N_i/\mathcal{F}_i) \to \Omega_b^\bullet(N_i, \mathcal{F}_i)$ for $i = 0$. Then the same holds for $i = 1$.*

Proof. Using Proposition 3.43, create the following commutative pentagon.

$$\begin{tikzcd}
 & M \arrow[dl, "p_0"'] \arrow[dr, "p_1"] & \\
N_0 \arrow[d, "\pi_0"'] & & N_1 \arrow[d, "\pi_1"] \\
N_0/\mathcal{F}_0 \arrow[rr, "\varphi"] & & N_1/\mathcal{F}_1.
\end{tikzcd}$$

All of φ^*, π_0^*, p_0^*, and p_1^* are isomorphisms of the relevant complexes. Thus π_1^* is too. $\square$

We could also make the same statement in cohomology.

4 BASIC FORMS

A singular foliation $\mathcal{F}$ of a manifold M is a partition of M into connected, weakly-embedded submanifolds, called leaves, of perhaps varying dimension, satisfying the following smoothness condition: for every $x \in M$ contained in a leaf L_x, there exist locally-defined vector fields X^i about x such that

- the X^i are tangent to the leaves;
- the X^i_x span T_xL_x

We then take the complex of basic differential forms to consist of those $\alpha \in \Omega^\bullet(M)$ such that

$$\iota_X\alpha = 0, \text{ and } \mathcal{L}_X\alpha = 0, \text{ for all } X \text{ tangent to the leaves.}$$

The set of $\mathcal{F}$-basic forms, denoted $\Omega^\bullet_b(M, \mathcal{F})$, is a de Rham subcomplex of differential forms. We relate this complex to one on the quotient (or leaf) space $M/\mathcal{F}$. While rarely a smooth manifold (for example, the leaf space of the irrational flow on the torus is not even Hausdorff), $M/\mathcal{F}$ is naturally a diffeological space; recall that diffeology is defined in Chapter 2. The quotient $\pi : M \to M/\mathcal{F}$ is diffeologically smooth, and $M/\mathcal{F}$ is equipped with a de Rham complex of diffeological differential forms, $\Omega^\bullet(M/\mathcal{F})$. Pullback by the quotient induces a one-to-one morphism from diffeological forms into basic forms. We seek singular foliations for which the following property holds.

Property (P). *The pullback $\pi^* : \Omega^\bullet(M/\mathcal{F}) \to \Omega^\bullet_b(M, \mathcal{F})$ is onto (hence, an isomorphism).*

Diffeology plays an important role. For example, we prove the foliation of the torus by an irrational flow has Property (P), and as previously mentioned its leaf space is not a manifold. More generally, we prove

(i) Regular foliations have Property (P) (Theorem 4.11).

(ii) $\mathcal{F}$ has Property (P) if the union of leaves of the same dimension is a diffeological submanifold (Theorem 4.20).

(iii) $\mathcal{F}$ has Property (P) if the induced singular foliation on $M \setminus \{x \mid \dim L_x = 0\}$ has Property (P) (Theorem 4.27).

We proved (i) independently, then found Hector, Marcías-Virgós, and Sanmartín-Carbón [HMVSC11] proved it earlier. We approach this problem using groupoid techniques. For a groupoid version of Property (P), replace $(M, \mathcal{F})$ with a Lie groupoid $G \rightrightarrows M$; replace $M/\mathcal{F}$ with M/G; and take the basic forms to be α such that $s^*\alpha = t^*\alpha$. Then, we arrive at (i) by proving

- Property (P) is a Morita-invariant;
- the holonomy groupoid of $\mathcal{F}$ has Property (P) if and only if $\mathcal{F}$ does;
- étale groupoids with countably generated pseudogroup (e.g. étale holonomy groupoids) have Property (P).

Hector et al. deal directly with the action of Haefliger's holonomy pseudogroup on a complete transversal. Our methods generalize their approach, and we make explicit some nontrivial correspondences untreated in [HMVSC11], specifically their Proposition 3.1.

Item (ii) has not appeared in the literature, and extends a few existing results. In [KW16], Karshon and Watts proved that, given a Lie group G acting on M, if its identity component G° acts properly, then corresponding action groupoid $G^\circ \ltimes M \rightrightarrows M$ has Property (P).

In [Wat22], Watts used [KW16] to prove that proper Lie groupoids have Property (P). In the source-connected case, we can give another proof of [KW16] and [Wat22]. It is by now well-established that proper Lie groupoids are linearizable, in the sense of Definition 4.22. Weinstein gave the first result of this type in [Wei02], where he showed, under some extra conditions, that linearizability at fixed points was sufficient for linearizability at finite type orbits (a term we do not define here). Zung made the next significant advance in [Zun06], by proving proper Lie groupoids, again satisfying some extra assumptions, are linearizable at their fixed points. Crainic and Struchiner collected these various results in [CS13], and gave a self-contained proof that proper Lie groupoids are linearizable without qualification, which we state as Theorem 4.25. By Proposition 4.24, linearizability implies the hypotheses of (ii) hold, hence the foliation of M by the orbits of G has Property (P). By source-connectedness, Proposition 4.7 implies this is equivalent to G having Property (P).

A key feature of the above arguments is that linearizability of a Lie groupoid is sufficient for Property (P) to hold. Since properness is sufficient but not necessary for linearizability, our class of groupoids for which (P) holds is distinct from those in [KW16] and [Wat22]. However, even linearizability is not necessary for Property (P). Our final result (iii) shows that the placement of the 0-dimensional leaves does not impact whether $\mathcal{F}$

has Property (P). For example, every singular foliation of $\mathbb{R}$ has Property (P), and many of these are not linearizable. More generally, given any singular foliation with Property (P), we may choose any closed subset C of M and declare its points to be 0-leaves, yielding a new singular foliation for which Property (P) also holds (by applying (iii), and then (ii)). This suggests Property (P) holds for many singular foliations. On the other hand, if the singular foliation has no 0-leaves, and we are not in the situation to apply (ii), then the results of this paper alone are not powerful enough to determine whether $\mathcal{F}$ has Property (P) directly. As of writing, we are not aware of any singular foliation for which Property (P) fails. We leave more discussion for the end of the chapter.

This chapter is structured as follows. In Section 4.1, we prove Theorem 4.11. In Section 4.2, we prove Theorems 4.20 and 4.25. In Section 4.3, we discuss some examples and future directions for research.

4.1 THE REGULAR CASE

In this section, we prove Theorem 4.11. In building up to this result, we recite and prove necessary tools relating the basic complex and the notions of Lie groupoids, Morita equivalence, holonomy pseudogroups, etc. As a reminder, we assume M is a Hausdorff and second countable manifold.

Definition 4.1 (Basic forms)**.** Fix a singular foliation $(M, \mathcal{F})$ with associated singular distribution Δ. A differential form $\alpha \in \Omega^\bullet(M)$ is *$\mathcal{F}$-basic* if for every section $X \in \Gamma_{\text{loc}}(\Delta)$,

$$\iota_X\alpha = 0, \text{ and } \mathcal{L}_X\alpha = 0. \tag{4.1}$$

The first condition says α is *horizontal*, and the second *invariant*. Denote the set of basic forms by $\Omega_b^\bullet(M, \mathcal{F})$.

Remark 4.2. Because $\mathcal{L}_X\alpha = \iota_X(d\alpha) + d(\iota_X\alpha)$, any horizontal form is invariant if and only if $\iota_X d\alpha = 0$. In other words, $\alpha \in \Omega_b^\bullet(M, \mathcal{F})$ if and only if $\iota_X\alpha = 0$ and $\iota_X d\alpha = 0$ for all $X \in \Gamma_{\text{loc}}(\Delta)$. But the interior derivative ι_X at x depends only on X_x. We therefore conclude that to prove α is $\mathcal{F}$-basic, it suffices to check (4.1) against any set of vector fields spanning Δ. In particular, given a Lie algebroid $(\mathfrak{g}, \rho)$, a form is $\mathfrak{g}$-basic (i.e. (4.1) holds for all $X \in \rho(\Gamma(\mathfrak{g}))$) if and only if it is basic with respect to the induced singular foliation.

We could potentially define, for any smooth singular distribution Δ, a Δ-basic form to be one which satisfies (4.1) for all $X \in \Gamma_{\text{loc}}(\Delta)$. While there are no technical problems with this definition, if Δ is not integrable, then by Corollary 3.23 there is no associated singular foliation $\mathcal{F}$, and hence no related leaf-space. Since our goal is to investigate how basic forms capture transverse structures, it therefore does not make sense for us to define basic forms in this generality.

The differential and exterior product of basic forms remain basic, so $\Omega^\bullet_b(M, \mathcal{F})$ is a subcomplex of $\Omega^\bullet(M)$. In the presence of a Lie groupoid, there is another complex of "basic" forms.

Definition 4.3 (Basic forms for groupoids)**.** Fix a Lie groupoid $G \rightrightarrows M$. A differential form $\alpha \in \Omega^\bullet(M)$ is G-basic if $s^*\alpha = t^*\alpha$. Denote the de Rham complex of these forms by $\Omega^\bullet_b(M, G)$.

For a Lie group G acting on M, there is also a notion of a G-basic form. This is a form α such that $g^*\alpha = \alpha$ for all $g \in G$, and $\iota_X\alpha$ for all X tangent to the orbits of G. However, this is not a fundamentally different notion than that from Lie groupoids: a form is G-basic if and only if it is $G \ltimes M$-basic (Lemma 3.3 in [Wat22]). We relate the complexes of G-basic and $\mathcal{F}_G$-basic forms in Proposition 4.7.

We now work to prove Theorem 4.11, or statement (i) from the Chapter introduction, namely that for a regular foliation $\mathcal{F}$, the pullback by the quotient $\pi : M \to M/\mathcal{F}$ is an isomorphism $\pi^* : \Omega^\bullet(M/\mathcal{F}) \to \Omega^\bullet_b(M, \mathcal{F})$. Another proof can be found in [HMVSC11], but we arrived at this result independently.

We outline the proof here. First, we use Proposition 4.6 to show the image of π^* is contained in the set of $\mathcal{F}$-basic forms, so the question is well-posed. Then, we invoke Proposition 4.7 to show that the $\mathcal{F}$-basic forms are exactly the Hol-basic forms, because Hol is a source-connected groupoid whose orbits are the leaves of $\mathcal{F}$. This reduces the question to whether $\pi^* : \Omega^\bullet(M/\mathcal{F}) \to \Omega^\bullet_b(M, \mathrm{Hol})$ is onto. Take a complete transversal $\iota : S \to M$ to $\mathcal{F}$, and obtain the étale holonomy groupoid $\mathrm{Hol}_S \rightrightarrows S$, which is Morita equivalent to $\mathrm{Hol} \rightrightarrows M$ by Lemma 2.41. Because of this equivalence, π^* is an isomorphism if and only if the pullback by $\pi_S : S \to S/\mathrm{Hol}_S$ is an isomorphism, by Proposition 4.8 and Corollary 4.9. But Hol_S has finitely generated pseudogroup by Lemma 2.41, and this implies π_S^* is an isomorphism by Lemma 4.10.

Before we prove Proposition 4.6, we need two results from diffeology. These auxilliary results appear as Articles 6.38 and 6.39 in [IZ13]. See there for the proofs.

Proposition 4.4. *Fix a diffeological space X with relation $\mathcal{R}$, and equip $X/\mathcal{R}$ with the quotient diffeology. The image of $\pi^* : \Omega^k(X/\mathcal{R}) \to \Omega^k(X)$ is the set of k-forms α satisfying either of the following, equivalent conditions*

(i) For any two plots $p, q : \mathsf{U} \to X$ with $\pi \circ p = \pi \circ p$, we have $\alpha(p) = \alpha(q)$. This is our main tool.

(ii) *In the pullback diagram*

$$\begin{tikzcd} X \times_\pi X \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathrm{pr}_2"] & X \arrow[d, "\pi"] \\ X \arrow[r, "\pi"] & X/\mathcal{R}. \end{tikzcd}$$

we have $\mathrm{pr}_1^* \alpha = \mathrm{pr}_2^* \alpha$*. In other words, the form* α *is basic with respect to the diffeological relation groupoid* $X \times_\pi X \rightrightarrows X$ *(used in the proof of Proposition 4.8).*

Lemma 4.5. *The pullback* $\pi^* : \Omega^k(X/\mathcal{R}) \to \Omega^k(X)$ *is injective.*

Now we state and prove the required Proposition.

Proposition 4.6.

(i) *For a singular foliation, every pullback of a form on* $M/\mathcal{F}$ *is* $\mathcal{F}$*-basic.*

(ii) *For a Lie groupoid, every pullback of a form on* M/G *is* G*-basic.*

Proof.

(i) Suppose $\beta \in \Omega^k(M/\mathcal{F})$, and set $\alpha := \pi^*\beta$. For any $X \in \Gamma_{\mathrm{loc}}(\Delta)$, we must show $\iota_X\alpha = 0$ and $\mathcal{L}_X\alpha = 0$. By Proposition 4.4, for any plots $p, q : \mathsf{U} \to M$ such that $\pi \circ p = \pi \circ q$, we have $p^*\alpha = q^*\alpha$. We can replace the domain U with any manifold. Thus, let $\Phi : \mathcal{D} \to M$ denote the flow of X (so $\mathcal{D} \subseteq \mathbb{R} \times M$ is open), and set $p := \Phi$, and $q := \mathrm{pr}_2$, the projection onto M. Since $\Phi^t(x) := \Phi(t, x)$ and $\mathrm{pr}_2(t, x) = x$ are on the same leaf, $\Phi^*\alpha = \mathrm{pr}_2^* \alpha$.

For $(t, x) \in \mathcal{D}$, we identify $T_{(t,x)}\mathcal{D}$ with $\mathbb{R} \oplus T_xM$. For every collection $v_1, \ldots, v_k \in T_xM$, at $t = 0$ we have

$$(\Phi^*\alpha)_{(0,x)}(1 \oplus v_1, v_2, \ldots, v_k) = \alpha_x(X_x + v_1, v_2, \ldots, v_k)$$
$$(\mathrm{pr}_2^* \alpha)_{(0,x)}(1 \oplus v_1, v_2, \ldots, v_k) = \alpha_x(v_1, v_2, \ldots, v_k).$$

This implies $\iota_{X_x}\alpha_x = 0$. We also have for each t, at $\vec{v} = (v_1, \ldots, v_k)$,

$$\alpha_x(\vec{v}) = (\mathrm{pr}_2^* \alpha)_{(x,t)}(\vec{v}) = (\Phi^*\alpha)_{(t,x)}(\vec{v}) = ((\Phi^t)^*\alpha)_x(\vec{v}),$$

hence

$$0 = \frac{d}{dt}\Big|_{t=0} \alpha_x(\vec{v}) = \frac{d}{dt}\Big|_{t=0} ((\Phi^t)^*\alpha)_x(\vec{v}) = (\mathcal{L}_X\alpha)_x(\vec{v}).$$

So we conclude both $\iota_X\alpha$ and $\mathcal{L}_X\alpha$ vanish.

(ii) This is Corollary 3.6 in [Wat22].

□

For this next proposition, we adapt the proof from [HS21], which deals only with regular foliations.

Proposition 4.7. *Let $G \rightrightarrows M$ be a Lie groupoid with associated singular foliation $\mathcal{F}$. Every G-basic form is $\mathcal{F}$-basic, and every $\mathcal{F}$-basic form is G°-basic.*

Proof. Let $(\mathfrak{g}, \rho)$ be the Lie algebroid of G, so $\mathfrak{g} = (\ker ds)|_M$ and $\rho = dt$. To prove a form is $\mathcal{F}$-basic, it suffices to test against vector fields in $\rho(\Gamma(\mathfrak{g}))$. For a section σ of $\mathfrak{g}$, denote the corresponding right-invariant vector field on G by $\tilde{\sigma}$. By a basic fact about Lie algebroids (page 122 in [Mei03]), $\tilde{\sigma}$ is s-related to 0, and also t-related to $\rho(\sigma)$. Therefore, for any form α on M,

$$\mathcal{L}_{\tilde{\sigma}} s^*\alpha = 0, \text{ and } \mathcal{L}_{\tilde{\sigma}} t^*\alpha = t^*\mathcal{L}_{\rho(\sigma)}\alpha. \tag{4.2}$$

Take the splitting $TG|_M = \mathfrak{g} \oplus TM$, which allows us to write

$$\begin{aligned} (s^*\alpha)_{1_x}(\xi_1 + v_1, \dots, \xi_k + v_k) &= \alpha_x(v_1, \dots, v_k) \\ (t^*\alpha)_{1_x}(\xi_1 + v_1, \dots, \xi_k + v_k) &= \alpha_x(\rho(\xi_1) + v_1, \dots, \rho(\xi_k) + v_k). \end{aligned} \tag{4.3}$$

- Suppose α is G-basic. Then $s^*\alpha = t^*\alpha$, and by setting $\xi_i = 0$ for $i > 1$ and $w_1 = 0$ in equation (4.3), we get

  $$\alpha_x(0, v_2, \dots, v_k) = \alpha_x(\rho(\xi_1), v_2, \dots, v_k) = (\iota_{\rho(\xi_1)}\alpha)_x(v_2, \dots, v_k).$$

  The left side is always 0, thus $\iota_{\rho(\sigma)}\alpha = 0$. For invariance, by equation (4.2),

  $$0 = \mathcal{L}_{\tilde{\sigma}} s^*\alpha = \mathcal{L}_{\tilde{\sigma}} t^*\alpha = t^*\mathcal{L}_{\rho(\sigma)}\alpha,$$

  and since t is a submersion, $\mathcal{L}_{\rho(\sigma)}\alpha = 0$. Therefore α is $\mathcal{F}$-basic.

- Now suppose α is $\mathcal{F}$-basic. The fact $\iota_{\rho(\sigma)}\alpha = 0$ implies

  $$\alpha_x(\rho(\xi_1) + v_1, \dots, \rho(\xi_k) + v_k) = \alpha_x(v_1, \dots, v_k),$$

  and by equation (4.3) we see $s^*\alpha = t^*\alpha$ at points $1_x \in M$. Furthermore, the assumption $\mathcal{L}_{\rho(\sigma)}\alpha = 0$ combined with equation (4.2) gives

  $$\mathcal{L}_{\tilde{\sigma}} s^* a = 0 = t^*\mathcal{L}_{\rho(\sigma)}\alpha = \mathcal{L}_{\tilde{\sigma}} t^*\alpha.$$

  Therefore $s^*\alpha$ and $t^*\alpha$ are invariant under the flows of the $\tilde{\sigma}$. These vector fields span $\ker ds$, an involutive subbundle of TG that foliates G by the connected components of the source-fibers. In particular, we can connect any arrow in the component 1_x to 1_x by travelling along the flows of the $\tilde{\sigma}$. Therefore $s^*\alpha = t^*\alpha$ on the union of connected components of the identity arrows. This is exactly the arrow space of $G^\circ \rightrightarrows M$, thus α is G°-basic.

$\square$

The following is Proposition 3.9 from [Wat22], but also found as Lemma 5.3.8 in [HS21].

Proposition 4.8. *Let $\mathcal{G}$ and $\mathcal{H}$ be Morita equivalent Lie groupoids, witnessed by an invertible bibundle $P : \mathcal{G} \to \mathcal{H}$. There is an isomorphism $P^* : \Omega^\bullet_b(H_0, \mathcal{H}) \to \Omega^\bullet_b(G_0, G)$ defined uniquely by the condition that $a_R^*\alpha = a_L^* P^*\alpha$ (where a_R and a_L are the anchor maps for the actions).*

Proof. The bibundle $P : G \to \mathcal{H}$ gives the following commutative diagram (see Definition 2.22).

$$\begin{tikzcd}
 & \mathcal{G} {}_s\times_{a_L} P \arrow[dl, "\mathrm{pr}_1"'] \arrow[dr, shift left, "\mu_L"] \arrow[dr, shift right, "\mathrm{pr}_2"'] & & P {}_{a_R}\times_t \mathcal{H} \arrow[dl, shift right, "\mu_R"'] \arrow[dl, shift left, "\mathrm{pr}_1"] \arrow[dr, "\mathrm{pr}_2"] & \\
\mathcal{G} \arrow[dr, shift left, "t"] \arrow[dr, shift right, "s"'] & & P \arrow[dl, "a_L"'] \arrow[dr, "a_R"] & & \mathcal{H} \arrow[dl, shift right, "s"'] \arrow[dl, shift left, "t"] \\
 & G_0 & & H_0 &
\end{tikzcd}$$

Let $\alpha \in \Omega^\bullet_b(H_0, \mathcal{H})$. The pullback $a_R^*\alpha$ is $P \times_{H_0} P$-basic (in the sense of Proposition 4.4 (ii)), because $a_R \circ \mathrm{pr}_1 = a_R \circ \mathrm{pr}_2$, for $\mathrm{pr}_i : P \times_{H_0} P \to P$. As $P \to H_0$ is a principal $\mathcal{G}$-bundle, by Remark 2.18, the Lie groupoid $P \times_{H_0} P$ is isomorphic to $\mathcal{G} \ltimes P$, so $a_R^*\alpha$ is $\mathcal{G} \ltimes P$-basic.

It is also $P \rtimes \mathcal{H}$-basic, because

$$\begin{aligned}
\mu_R^* a_R^* \alpha &= \mathrm{pr}_2^*\, s^*\alpha && \text{by commutativity of the diagram} \\
&= \mathrm{pr}_2^*\, t^*\alpha && \text{because } \alpha \text{ is } \mathcal{H}\text{-basic} \\
&= \mathrm{pr}_1^*\, a_R^*\alpha && \text{by commutativity of the diagram}
\end{aligned}$$

Exactly like above, we conclude $a_R^*\alpha$ is $P \times_{G_0} P$-basic. By Proposition 4.4 (ii), this is equivalent to $a_R^*\alpha = a_L^*\beta$, for some $\beta \in \Omega^\bullet(G_0)$. Note that β is unique because a_L is a surjective submersion (for instance, we could use Lemma 4.5). We denote $P^*\alpha := \beta$, and claim β is G-basic. This is a computation.

$$\begin{aligned}
\mathrm{pr}_1^* s^*\beta &= \mathrm{pr}_2^* a_L^*\beta && \text{by commutativity of the diagram}\\
&= \mathrm{pr}_2^* a_R^*\alpha && \text{by choice of } \beta\\
&= \mu_L^* a_R^*\alpha && \text{because } a_R^*\alpha \text{ is } \mathcal{G} \ltimes P\text{-basic}\\
&= \mu_L^* a_L^*\beta && \text{by choice of } \beta\\
&= \mathrm{pr}_1^* t^*\beta && \text{by commutativity of the diagram.}
\end{aligned}$$

Because pr_1 is a surjective submersion, $s^*\beta = t^*\beta$. Therefore P^* is well-defined, and it is evidently a homomorphism of complexes. Its inverse is $(P^{-1})^*$, so it is an isomorphism. This completes the proof. □

Corollary 4.9. *Assume $\mathcal{G}$ and $\mathcal{H}$ are Morita equivalent Lie groupoids, and let $\pi_{\mathcal{G}} : G_0 \to G_0/\mathcal{G}$ and $\pi_{\mathcal{H}} : H_0 \to H_0/\mathcal{H}$ denote the quotient maps. The pullback π_G^* surjects onto G-basic forms if and only if $\pi_{\mathcal{H}}^*$ is onto $\mathcal{H}$-basic forms.*

Proof. Take an invertible bibundle $P : \mathcal{G} \to \mathcal{H}$ as in Proposition 4.8. The map $\Psi : H_0/\mathcal{H} \to G_0/G$ defined by $\pi_{\mathcal{H}}(y) \mapsto \pi_G(a_L(a_R^{-1}(y)))$ is a well-defined diffeological diffeomorphism by Proposition 2.32. Now assume π_G^* is onto basic forms, and take $\alpha \in \Omega_b^\bullet(M, \mathcal{H})$. Set $\beta := P^*\alpha$. By our assumption on $\pi_{\mathcal{G}}^*$, there is some $\overline{\beta} \in \Omega^\bullet(G_0/G)$ with $\pi_G^*\overline{\beta} = \beta$. Then, because $\pi_{\mathcal{G}} \circ a_L = \Psi \circ \pi_{\mathcal{H}} \circ a_R$,

$$a_L^*\beta = a_L^*\pi_G^*\overline{\beta} = a_R^*(\pi_{\mathcal{H}}^*\Psi^*\overline{\beta}).$$

But $a_L^*\beta = a_R^*\alpha$, by definition of β (see the previous proof), and since a_R is a surjective submersion, we get $\alpha = \pi_{\mathcal{H}}^*\Psi^*\overline{\beta}$. In other words, $\pi_{\mathcal{H}}^*$ also surjects onto basic forms. For the converse direction, work with $(P^{-1})^*$. □

Now we have a final, key, lemma.

Lemma 4.10. *Let $G \rightrightarrows M$ be an étale Lie groupoid with countably generated associated pseudogroup $\Psi(G)$. Then the pullback by $\pi : M \to M/G$ is onto G-basic forms.*

Proof. Let $\alpha \in \Omega_b^\bullet(M, G)$. Take $p, q : \mathsf{U} \to M$ such that $\pi \circ p = \pi \circ q$. To show α is in the image of π^*, by Proposition 4.4 (i) it suffices to show $p^*\alpha = q^*\alpha$. First, note that α is $\Psi(G)$-invariant, since for any $f = t \circ \sigma \in \Psi(G)$, we have

$$f^*\alpha = \sigma^* t^*\alpha = \sigma^* s^*\alpha = \mathrm{id}^*\alpha = \alpha.$$

Say $\{f_i\}_{i=1}^\infty$ generates $\Psi(G)$. For each tuple $I := (i_1, \dots, i_N)$, define $f_I := f_{i_1} \circ \cdots \circ f_{i_N}$, and set $C_I := \{r \in \mathsf{U} \mid f_I(p(r)) = q(r)\}$. Each C_I is closed in U, and we claim $\mathsf{U} \subseteq \bigcup_I C_I$ (hence equality holds). Indeed, for any $r \in \mathsf{U}$, we have $\pi(p(r)) = \pi(q(r))$, so there is an arrow $p(r) \mapsto q(r)$. Taking σ to be a local section of s such that $\sigma(p(r))$ is this arrow, then we can take $f := t \circ \sigma \in \Psi(G)$ such that $f(p(r)) = q(r)$. Using our generating family for $\Psi(G)$, we can write $f = f_I$ locally near r for some I, hence $r \in C_I$.

By the Baire category theorem, $\bigcup_I \operatorname{int}(C_I)$ is open and dense in U. But on each $\operatorname{int}(C_I)$, we have $f_I \circ p = q$, so by $\Psi(G)$-invariance of α,

$$p^*\alpha = p^* f^* \alpha = (f_I \circ p)^* \alpha = q^* \alpha.$$

As this holds on the open dense subset $\bigcup_I \operatorname{int}(C_I)$, by continuity $p^*\alpha = q^*\alpha$ on all of U, as required. □

We may now give the formal statement and proof of our first main result.

Theorem 4.11. *Suppose $(M, \mathcal{F})$ is a regular foliation. Equip M and $M/\mathcal{F}$ with their manifold and quotient diffeology, respectively. The quotient map $\pi : M \to M/\mathcal{F}$ is diffeologically smooth, and its pullback restricts to an isomorphism from diffeological forms on $M/\mathcal{F}$ to $\mathcal{F}$-basic forms on M. In other words, $\pi^* : \Omega^\bullet(M/\mathcal{F}) \to \Omega^\bullet_b(M, \mathcal{F})$ is an isomorphism (cf. [HMVSC11]).*

Recall that we must work with diffeology, because $M/\mathcal{F}$ is generally not a smooth manifold. For instance, the leaf space of the irrational flow on the torus is not even Hausdorff.

Proof. The pullback π^* is injective by Lemma 4.5, and maps into basic forms by Proposition 4.6 (i). It remains is to show π^* is surjective. The foliation $\mathcal{F}$ is induced by its holonomy groupoid $\operatorname{Hol} \rightrightarrows M$. As Hol is source-connected, by Proposition 4.7 the $\mathcal{F}$-basic and Hol-basic forms coincide. Since $M/\operatorname{Hol} = M/\mathcal{F}$, it therefore suffices to show that π^* is onto Hol-basic forms.

Fix a complete transversal $\iota : S \to M$ to $\mathcal{F}$. The restriction $\operatorname{Hol}_S \rightrightarrows S$ of Hol to S is Morita equivalent to Hol, by Lemma 2.41. Therefore, by Corollary 4.9, to show π^* surjects onto Hol-basic forms, we may instead show the pullback by $\pi_S : S \to S/\operatorname{Hol}_S$ is onto Hol_S-basic forms. The groupoid $\operatorname{Hol}_S \rightrightarrows S$ is étale, and its associated pseudogroup is countably generated, by Lemma 2.41. So, we may apply Lemma 4.10 to complete the proof. □

As mentioned in the Introduction, this result was also proved by Hector et al. in [HMVSC11], though we only found their work after ours was completed. In our argument, we have given a complete account of the correspondence between $\Psi(\operatorname{Hol}_S)$-invariant forms on S, and basic forms on M. This correspondence is also used in [HMVSC11], as their Proposition

3.1, but they do not give a proof. In particular, our argument shows the role Morita-invariance plays in establishing this correspondence naturally, and this is not noted in [HMVSC11]. Generally, our techniques provide us the framework to prove more general facts about the basic complex of some Lie groupoids; for instance, we end with the following corollary.

Corollary 4.12. *If $G \rightrightarrows M$ is a source-connected Lie groupoid Morita equivalent to an étale groupoid (sometimes called a foliation groupoid), the pullback by $\pi : M \to M/G$ is an isomorphism onto G-basic forms.*

Proof. Because G is Morita equivalent to an étale groupoid, its associated isotropy groups are discrete (Proposition 5.20 in [MM03]). Therefore the singular foliation $\mathcal{F}$ associated to G is regular, and Theorem 4.11 every $\mathcal{F}$-basic form is a pullback from the quotient. But the $\mathcal{F}$-basic forms are exactly the G-basic forms by source-connectedness and Proposition 4.7, so the pullback must be onto G-basic forms as well. □

4.2 THE SINGULAR CASE

We now prove Theorems, 4.20, and 4.27, or in other words statements (ii) and (iii) from the Chapter introduction. We begin by establishing some terminology, then prove the theorem, and afterwards give a key example. We finish by extending the techniques in the proof of Theorem 4.20 to get Theorem 4.27, or statement (iii) in the Introduction, which handles a broader class of singular foliations, in the sense of Example 4.17.

We use the following notation.

Notation 4.13. Given a singular distribution Δ, we denote by M_{*k}, for $* \in \{=, \geq, >, <, \leq, \neq\}$, the set $M_{*k} := \{x \in M \mid \dim \Delta_x * k\}$.

Remark 4.14. As a consequence of the definition of a smooth singular distribution, the map $x \mapsto \dim \Delta_x$ is lower semi-continuous, and therefore the sets $M_{\geq k}$ are open in M.

Definition 4.15 (Decomposition by dimension)**.** A singular foliation $(M, \mathcal{F})$ is *decomposed by dimension* if the sets $M_{=k}$, consisting of points in leaves of dimension k, are diffeological submanifolds of M, perhaps with components of varying dimension.

Remark 4.16. An arbitrary singular foliation is not necessarily decomposed by dimension. For instance, the singular foliation of $\mathbb{R}$ whose 0-leaves are the points of some closed set C, and whose 1-leaves are the components of $\mathbb{R} \smallsetminus C$, is not decomposed by dimension. By taking products of foliations, we can generate examples of singular foliations for which $M_{=k}$ is not a diffeological submanifold, for various k.

Example 4.17.

- If a foliation consists of leaves of only two dimensions, $k_1 < k_2$, then because $M_{=k_2}$ is open (Remark 4.14), we need only check if $M_{=k_1}$ is a diffeological submanifold. Examples arising in this way include those singular foliations induced by: the action of the special linear group SL_n on $n \times n$ matrices M_n; the action of the (potentially) indefinite orthogonal group $O(p,q)$ on $\mathbb{R}^{p+q}$; and the action of $\mathbb{R}$ on $\mathbb{R}^n$ by the flow of the Euler vector field. In all cases, $k_1 = 0$ and $M_{=0}$ has one member, the origin.

- In Section 4.3, we will see an entire class of singular foliations decomposed by dimension coming from certain Lie groupoids, namely, the linearizable (in particular, proper; see Theorem 4.25) ones.

- A *Riemannian* singular foliation is a singular foliation $\mathcal{F}$ of a Riemannian manifold (M,g), such that if a geodesic is perpendicular to a leaf at one point, it remains perpendicular to all the leaves. These were introduced by Molino [Mol88]. For a more recent survey, see [ABT13]. Molino proved that Riemannian singular foliations are decomposed by dimension, see [Mol88, Chapter 6]. In this setting, Mendes and Radeschi [MR19] also discuss the basic complex, although they do not use diffeology.

Take an arbitrary singular foliation $(M,\mathcal{F})$ with associated distribution Δ. For $* \in \{=,\geq,>,<,\leq,\neq\}$, set

$$\mathcal{F}_{*k} := \{L \in \mathcal{F} \mid \dim L * k\}.$$

Lemma 4.18. *If M_{*k} is a diffeological submanifold of M, then $(M_{*k},\mathcal{F}_{*k})$ is a singular foliation.*

Proof. By the Stefan-Sussmann Theorem 3.20, we may take a collection of arrows $\mathcal{A}$ such that the orbits of $\Psi\mathcal{A}$ are the leaves of $\mathcal{F}$. Let $\mathcal{A}'$ consist of the arrows in $\mathcal{A}$ restricted to M_{*k}. Because M_{*k} is a diffeological submanifold, this is a collection of arrows, and so by Stefan's Theorem 3.14, the orbits of $\Psi\mathcal{A}'$ form a singular foliation of M_{*k}. But these orbits are exactly the elements of $\mathcal{F}_{*k}$, hence $\mathcal{F}_{*k}$ is indeed a singular foliation. □

Lemma 4.19. *If M_{*k} is a diffeological submanifold of M, and $\alpha \in \Omega^\bullet_b(M,\mathcal{F})$, then $\alpha|_{M_{*k}}$ is $\mathcal{F}_{*k}$-basic.*

Proof. Take $\mathcal{A}$ and $\mathcal{A}'$ as in the previous lemma. Then α is $\mathcal{F}$-invariant, hence $\Psi\mathcal{A}$-invariant. This means $\alpha' := \alpha|_{M_{*k}}$ is $\Psi\mathcal{A}'$-invariant, hence $\mathcal{F}_{*k}$-

invariant. As for horizontal, we must take a slightly pedantic approach. Denote the inclusions

$$\begin{tikzcd} L \arrow[r, hook, "\iota'"] \arrow[rr, hook, bend left, "\iota''"] & M_{*k} \arrow[r, hook, "\iota"] & M \end{tikzcd}$$

Both ι' and ι'' are smooth immersions, since L is weakly-embedded, but ι is merely smooth. Suppose $v \in T_xL$. We want to show $\iota_{\iota'_* v}\alpha' = 0$. Compute

$$\iota_{\iota'_* v}\alpha' = \alpha'(\iota'_* v, \cdot) = \alpha(\iota_* \iota'_* v, \iota_* \cdot) = \alpha(\iota''_* v, \iota_* \cdot).$$

But the right side is 0, because α is $\mathcal{F}$-horizontal. □

Now we come to the main result.

Theorem 4.20. *Suppose the singular foliation $(M, \mathcal{F})$ is decomposed by dimension. Equip M and $M/\mathcal{F}$ with the manifold and quotient diffeology, respectively. The quotient map $\pi : M \to M/\mathcal{F}$ is diffeologically smooth, and pulling back by the quotient map is an isomorphism from diffeological forms on $M/\mathcal{F}$ to $\mathcal{F}$-basic forms on M. In other words, $\pi^* : \Omega^\bullet(M/\mathcal{F}) \to \Omega^\bullet_b(M, \mathcal{F})$ is an isomorphism.*

Proof. By Proposition 4.6, π^* maps into $\mathcal{F}$-basic forms, and by Lemma 4.5, π^* is injective. It remains to show π^* is surjective. For $* \in \{\geq, =\}$, equip M_{*k} with the singular foliation $\mathcal{F}_{*k}$: in the $\geq$ case, the set $M_{\geq k}$ is open (and this does not require any assumption on $\mathcal{F}$, by Remark 4.14), hence is a diffeological submanifold; in the $=$ case, we are assuming $M_{=k}$ is a diffeological submanifold. Fix $\alpha \in \Omega^\bullet_b(M, \mathcal{F})$. Consider the statement

$$\alpha|_{M_{\geq k}} \text{ is the pullback of a form on the quotient } M_{\geq k}/\mathcal{F}_{\geq k}. \qquad S(k)$$

Let $k_{\max}$ denote the highest dimension of the leaves of $\mathcal{F}$. If $S(k_{\max})$ holds, and if $S(k+1) \implies S(k)$, then $S(0)$ holds, which is what we want to prove. Now, $S(k_{\max})$ is equivalent to: there is a form β on $M_{=k_{\max}}/\mathcal{F}_{=k_{\max}}$ such that $\pi^*\beta = \alpha|_{M_{=k_{\max}}}$. But $(M_{=k_{\max}}, \mathcal{F}_{=k_{\max}})$ is a regular foliation, and $\alpha|_{M_{=k_{\max}}}$ is $\mathcal{F}_{=k_{\max}}$-basic by Lemma 4.19, so $S(k_{\max})$ holds by Theorem 4.11.

Now assume $S(k+1)$. We will use Proposition 4.4 (i) to conclude $S(k)$. Let $p, q : \mathsf{U} \to M_{\geq k}$ be plots such that $\pi \circ p = \pi \circ q$. Recall here that $M_{\geq k}$ is open, and $M_{\geq k+1}$ is open in $M_{\geq k}$ (equivalently, open in M). Set

$$A := p^{-1}(M_{\geq k+1}) \quad (= q^{-1}(M_{\geq k+1})), \text{ which is open in } \mathsf{U} \text{ because } p \text{ is continuous.}$$
$$B := p^{-1}(M_{=k}) \quad (= q^{-1}(M_{=k})).$$

Then $\mathsf{U} = A \sqcup B$, and by an exercise in point-set topology, $\mathsf{U} = \overline{A} \cup \text{int}(B)$. We will show $\alpha(p) = \alpha(q)$ first on A, and then on $\text{int}(B)$. By continuity, this yields $\alpha(p) = \alpha(q)$ on $\overline{A} \cup \text{int}(B) = \mathsf{U}$.

- For A: The plots p and q restrict to maps $p', q' : A \to M_{\geq k+1}$, which are smooth maps between the U-open set A and the M-open set $M_{\geq k+1}$ because they are restrictions to open sets of the smooth maps p and q. We have $\pi \circ p' = \pi \circ q'$, and we are assuming $S(k+1)$. Therefore by Proposition 4.4 (i),

$$\alpha|_{M_{\geq k+1}}(p') = \alpha|_{M_{\geq k+1}}(q'), \text{ which implies } \alpha(p)|_A = \alpha(q)|_A.$$

- For $\operatorname{int}(B)$: We may assume $\operatorname{int}(B)$ is non-empty. Fix $r \in \operatorname{int}(B)$. Because the points $p(r)$ and $q(r)$ are on the same leaf, by the Stefan-Sussmann Theorem 3.20, there is a diffeomorphism $\xi : V \to V'$ (In the notation of Section 3.1, $\xi \in \Psi\mathcal{A}$), from a neighbourhood V of $p(r)$ to a neighbourhood V' of $q(r)$, that preserves the leaves, and sends $p(r)$ to $q(r)$. Let U' be an open, connected neighbourhood of r such that $\mathsf{U}' \subseteq \operatorname{int}(B) \cap p^{-1}(V)$. Define

$$\begin{aligned} p' &: \mathsf{U}' \to M_{=k}, \quad r' \mapsto \xi(p(r')) \\ q' &: \mathsf{U}' \to M_{=k}, \quad r' \mapsto q(r'). \end{aligned}$$

Since we assume $M_{=k}$ is a diffeological submanifold of M, and p', q' are smooth as maps $\mathsf{U}' \to M$, they are smooth maps $\mathsf{U}' \to M_{=k}$. In particular, they are continuous. As U' is connected, its images under p' and q' are connected, and must lie in a connected component of $M_{=k}$. Because $p'(r) = q'(r)$, in fact both p' and q' map into a single connected component, say $M^\circ_{=k}$, of $M_{=k}$. By abuse of notation, denote the restrictions of p' and q' to maps $\mathsf{U}' \to M^\circ_{=k}$ again by p' and q'. By definition of the subset diffeology on $M^\circ_{=k}$, both p' and q' are smooth maps $\mathsf{U}' \to M^\circ_{=k}$. Observe that

$$\begin{aligned} \pi(p'(r')) &= \pi(\xi(p'(r'))) && \text{by definition of } p' \\ &= \pi(p'(r')) && \text{because } \pi \text{ is } \xi\text{-invariant} \\ &= \pi(q'(r')) && \text{because } \pi \circ p' = \pi \circ q', \end{aligned}$$

so $\pi \circ p' = \pi \circ q'$.

Therefore, we have a regular foliation $(M^\circ_{=k}, \mathcal{F}_{=k})$ (discarding the leaves of $\mathcal{F}_{=k}$ in other connected components), two plots p', q' of $M^\circ_{=k}$ with $\pi \circ p' = \pi \circ q'$, and furthermore, by Lemma 4.19, the $\mathcal{F}_{=k}$-basic form $\alpha|_{M^\circ_{=k}}$ on $M^\circ_{=k}$. By Theorem 4.11, $\alpha|_{M^\circ_{=k}}$ is the pullback of some form on $M^\circ_{=k}/\mathcal{F}_{=k}$. By Proposition 4.4, we get

$$\alpha|_{M^\circ_{=k}}(p') = \alpha|_{M^\circ_{=k}}(q'), \text{ which implies } \alpha(\xi \circ p)|_{U'} = \alpha(q)|_{U'}.$$

Because α is $\mathcal{F}$-basic and ξ preserves the leaves, $\alpha(\xi \circ p) = \alpha(p)$. As r was arbitrary, we can conclude that $\alpha(p)|_{\mathrm{int}(B)} = \alpha(q)|_{\mathrm{int}(B)}$.

Therefore, for any two plots p, q of M such that $\pi \circ p = \pi \circ q$, we have proved $\alpha(p) = \alpha(q)$. By Proposition 4.4 (i), α is the pullback of some diffeological form on $M/\mathcal{F}$. $\square$

4.3 EXAMPLES AND NEXT STEPS

Here we will work out an example amenable to Theorem 4.20, but not handled by any of the existing results [HMVSC11] (the associated foliation is not regular), [Wat22], or [KW16] (the action is not proper).

Example 4.21. Fix n. Consider special linear group SL_n acting on the $n \times n$ matrices M_n from the left. Denote the quotient diffeological space M_n / SL_n by X, with quotient map π. Call the determinant map det. We also introduce the map

$$\sigma : \mathbb{R} \to M_n, \quad \alpha \mapsto (A_{ij}) \text{ where } A_{ij} = \begin{cases} \alpha & \text{if } i = j = 1 \\ 1 & \text{if } i = j \neq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Write ι for the composition $\pi \circ \sigma$. Because det is SL_n-invariant, it factors through π, meaning there exists a map $c : X \to \mathbb{R}$ making the diagram below commute:

$$\begin{tikzcd} & M_n \arrow[dl, "\det"'] \arrow[d, "\pi"] & \\ \mathbb{R} & X \arrow[l, "c"] & \mathbb{R} \arrow[l, "\iota"] \arrow[ul, "\sigma"'] \end{tikzcd}$$

The map c is smooth because π is a quotient map, and det is smooth. The map ι is smooth because σ is a lift. We claim that $c^* : \Omega^\bullet(\mathbb{R}) \to \Omega^\bullet(X)$ is an isomorphism.

- Injective: since $\mathrm{id}_{\mathbb{R}} = \det \circ \sigma = c \circ \iota$, then $\mathrm{id}_{\Omega^k(\mathbb{R})} = \mathrm{id}_{\mathbb{R}}^* = \iota^* \circ c^*$, so c^* has a left-inverse, and is injective.

- Surjective: for $\alpha \in \Omega^k(X)$, we claim $c^*(\iota^*\alpha) = \alpha$. By Lemma 4.5, π^* is injective, so it suffices to show $\pi^* c^* \iota^* \alpha = \pi^* \alpha$. First, observe $\iota \circ c \circ \pi = \pi \circ \sigma \circ \det$, and so we must prove

$$(\pi \circ \sigma \circ \det)^* \alpha = \pi^* \alpha. \tag{4.4}$$

Note that det restricts and descends to an isomorphism $\mathrm{GL}_n / \mathrm{SL}_n \to \mathbb{R} \setminus \{0\}$. Therefore, for $A \in \mathrm{GL}_n$, both $\sigma(\det A)$ and A are in the same SL_n-orbit. In particular, $(\pi \circ \sigma \circ \det)|_{\mathrm{GL}_n} = \pi|_{\mathrm{GL}_n}$, and so (4.4) holds on the subset GL_n of M_n. But GL_n is dense in M_n, hence (4.4) must hold on all of M_n, as required.

We note that the argument given here differs from that in [Miy23a, Example 5.17]. There, we erroneously assumed that if $c^*(\iota^*\alpha)$ and α coincide on the open dense subset $\iota(\mathbb{R})$ of X, then they agree on X. While it is true that diffeological differential forms are sections of a bundle over X (see [IZ13, Article 6.50]), the fact X is not Hausdorff means two smooth sections of this bundle might agree on an open dense subset without necessarily coinciding on all of X. The argument we provide here fixes this error.

The group SL_n is connected, and the induced foliation is decomposed by dimension, by Example 4.17. By Theorem 4.20, we see that π^* is an isomorphism. Therefore, $\det^* = \pi^* \circ c^*$ is an isomorphism from $\Omega^\bullet(\mathbb{R})$ to $\Omega^\bullet_b(M_n, \mathrm{SL}_n)$. In particular, for example, we have proven that the only SL_n-invariant smooth functions $M_n \to \mathbb{R}$ are those of the form $f \circ \det$, for $f \in C^\infty(M_n)$. Another consequence is that the basic cohomology for the SL_n action is isomorphic to the de Rham cohomology of $\mathbb{R}$.

Example: linearizable Lie groupoids

A rich source of singular foliations that are decomposed by dimension are those induced by "linearizable" Lie groupoids. By a theorem of Crainic and Struchiner [CS13], given here as Theorem 4.25, every proper Lie groupoid is linearizable. These include, for instance, actions of compact Lie groups.

First we review the linearization of a Lie groupoid about an orbit. Our sources are [CS13] and [Fer15]. Fix a Lie groupoid $G \rightrightarrows M$, and orbit $\mathcal{O}$ through $x \in M$. We can form the restricted groupoid $G_{\mathcal{O}} \rightrightarrows \mathcal{O}$, whose arrows are those arrows in G which begin (and end) in $\mathcal{O}$, and all the structure maps are induced from $G \rightrightarrows M$. This is a Lie groupoid.

Let $\nu(\mathcal{O})$ denote the normal bundle over $\mathcal{O}$, and similarly denote $\nu(G_{\mathcal{O}})$. We specify the Lie groupoid $\nu(G_{\mathcal{O}}) \rightrightarrows \nu(\mathcal{O})$ using the short exact sequence of groupoids:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & T\mathcal{G}_{\mathcal{O}} & \longrightarrow & T\mathcal{G} & \longrightarrow & \nu(\mathcal{G}_{\mathcal{O}}) & \longrightarrow & 1 \\
 & & \downdownarrows & & \downdownarrows & & \downdownarrows & & \\
0 & \longrightarrow & T\mathcal{O} & \longrightarrow & TM & \longrightarrow & \nu(\mathcal{O}) & \longrightarrow & 0.
\end{array}$$

Definition 4.22. A Lie groupoid is *linearizable at the orbit* $\mathcal{O}$ if there is an open neighbourhood U of $\mathcal{O} \subseteq M$ and an open neighbourhood V of $\mathcal{O} \subseteq$

$\nu(\mathcal{O})$ (viewing $\mathcal{O}$ as the image of the zero-section), and an isomorphism of the Lie groupoids $G|_U \rightrightarrows U$ and $\nu(G_{\mathcal{O}})|_V \rightrightarrows V$, which is the identity on $G_{\mathcal{O}} \rightrightarrows \mathcal{O}$. A Lie groupoid that is linearizable at every orbit is *linearizable*.

Remark 4.23. While the above definition is convenient to write, we will need some different descriptions of the linear model. Recall that we denote the isotropy Lie group at x by $G_x := s^{-1}(x) \cap t^{-1}(x)$ (and we do not use caligraphic font for isotropy groups).

(i) The groupoid $G_{\mathcal{O}}$ acts on $\nu(\mathcal{O})$ from the left, with anchor π by,

$$\mu_L(g, [v]) = g \cdot [v] := [dt_g(\tilde{v})] \text{ where } \tilde{v} \in T_g G \text{ satisfies } ds_g(\tilde{v}) = v.$$

The action is well-defined, and yields the associated action groupoid $G_{\mathcal{O}} \ltimes \nu(\mathcal{O})$. One can check this is isomorphic to $\nu(G_{\mathcal{O}})$, via $[\tilde{v}_g] \mapsto (g, [ds_g \tilde{v}])$.

(ii) Consider the principal G_x-bundle $t : P_x \to \mathcal{O}$. The action μ_L from (i) provides a left action of G_x on $\nu_x\mathcal{O}$. Then, we can form the associated bundles $P_x \times_{G_x} \nu_x\mathcal{O}$. This bundle is isomorphic to $\nu(\mathcal{O})$ via $[k, w] \mapsto k \cdot w$. Under this identification, we obtain the Lie groupoid $\mathcal{G}_{\mathcal{O}} \ltimes \nu(\mathcal{O}) \rightrightarrows P_x \times_{G_x} \nu_x\mathcal{O}$, which is isomorphic to $\nu(\mathcal{G}_{\mathcal{O}})$.

Now we state and prove the assertion that a singular foliation induced by a linearizable Lie groupoid is decomposed by dimension. Note that this is implicitly proved for proper groupoids by Posthuma, Tang, and Wang in [PTW21] (their Proposition 3.4). But because linearizable Lie groupoids are a key example for us, and we can avoid assuming properness, we give a self-contained but not fundamentally dissimilar proof.

Proposition 4.24. *A singular foliation induced by a linearizable Lie groupoid is decomposed by dimension.*

Proof. Call the Lie groupoid $\mathcal{G} \rightrightarrows M$. Fix an orbit $\mathcal{O}$ of $\mathcal{G}$, say its codimension is k (it is convenient to take codimension instead of dimension here), and fix $x \in \mathcal{O}$. By definition of linearizability, there exists an open neighbourhood U of $\mathcal{O}$ in M, an open neighbourhood V of $\mathcal{O}$ in $\nu(\mathcal{O})$, and an isomorphism of the Lie groupoids $G|_U \rightrightarrows U$ and $(G_{\mathcal{O}} \ltimes \nu(\mathcal{O}))|_V \rightrightarrows V$. Since the dimension of the orbit about $y \in U$ is determined by $\dim G_y$, we will first relate G_y to G_x.

Say $y \in U \subseteq M$ corresponds to $v \in V \subseteq \nu(\mathcal{O})$. By Remark 4.23 (ii), we view $\nu(\mathcal{O})$ as $P_x \times_{G_x} \nu_x\mathcal{O}$. Then there is a unique $[k, w]$ in this associated bundle such that $k \cdot w = v$. We claim

the projection $\mathcal{G}_{\mathcal{O}} \ltimes \nu(\mathcal{O}) \to \mathcal{G}_{\mathcal{O}}$ restricts to a diffeomorphism $\mathrm{iso}(v) \to k\, \mathrm{stab}_{G_x}(w) k^{-1}$.

Here $\mathrm{iso}(v)$ is the isotropy group at v, i.e. the collection of arrows in the action groupoid $\mathcal{G}_{\mathcal{O}} \ltimes \nu(\mathcal{O}) \rightrightarrows \nu(\mathcal{O})$ from v to itself, and $\mathrm{stab}_{G_x}(w)$ is the stabilizer of $w \in \nu(\mathcal{O})$ with respect to the action of G_x on $\nu(\mathcal{O})$.

- Well-defined: in the action groupoid $\mathcal{G}_{\mathcal{O}} \ltimes \nu(\mathcal{O})$, the only arrows with source v are of the form (g, v), because the source map is the projection of the second coordinate. Also, by definition of the action, the arrows g and k in $\mathcal{G}$ must satisfy

  $$g : \pi(v) \mapsto \pi(g \cdot v) = \pi(v)$$
  $$k : x \mapsto \pi(k \cdot w) = \pi(v),$$

  Hence $k^{-1}gk : x \mapsto x$, and is an element of G_x. Furthermore

  $$k^{-1}gk \cdot w = k^{-1}g \cdot v = k^{-1} \cdot v = w,$$

  so $k^{-1}gk$ is in $\mathrm{stab}_{G_x}(w)$. Therefore $g = k(k^{-1}gk)k^{-1}$ is in $k\,\mathrm{stab}_{G_x}(w)k^{-1}$, as required.

- Smooth: the projection is smooth, and we restrict its domain and codomain to embedded submanifolds, hence the restriction is smooth.

- Inverse: the inverse map $g \mapsto (g, v)$ is well-defined because $g = k\gamma k^{-1}$ for some $\gamma \in \mathrm{stab}_{G_x}(w)$, hence

  $$g \cdot v = k\gamma k^{-1} \cdot v = k\gamma \cdot w = k \cdot w = v.$$

  It is smooth because it is the restriction to embedded submanifolds of the inclusion $\mathcal{G}_{\mathcal{O}} \to \mathcal{G}_{\mathcal{O}} \ltimes \nu(\mathcal{O})$.

Therefore, $G_y \cong \mathrm{iso}(v) \cong k\,\mathrm{stab}_{G_x}(w)k^{-1}$. In particular, $\dim G_y = \dim \mathrm{stab}_{G_x}(w)$; and $\dim G_y = \dim G_x$ if and only if $\mathrm{stab}_{G_x}(w)$ is an open submanifold of G_x that contains the identity. In this case, because $\mathrm{stab}_{G_x}(w)$ is a subgroup of G_x, necessarily $\mathrm{stab}_{G_x}(w) \supseteq G_x^\circ$, where G_x° is the identity component of G_x. In other words, the set of points $w \in V \subseteq \nu(\mathcal{O})$ fixed by G_x° corresponds exactly to the set of $y \in U \subseteq M$ such that $\dim G_y = \dim G_x$.

Denote by $(\nu_x\mathcal{O})^{G_x^\circ}$ the vector subspace of fixed points of G_x°, and recall $\mathrm{codim}(\mathcal{O}) = \dim G_x = k$. We conclude that the diffeomorphism $U \to V$ from the linearization descends to a bijection $U \cap M_{=k} \to V \cap P_x \times_{G_x} (\nu_x\mathcal{O})^{G_x^\circ}$. The codomain is an embedded submanifold of V, so we may take these bijections as charts for an atlas of $U \cap M_{=k}$. As we choose different $x \in M_{=k}$, the resulting smooth structures on $U \cap M_{=k}$ may have different dimensions. Thus, we can only conclude that each connected (in the subspace topology) component of $M_{=k}$ is an embedded manifold. But this is the requirement to be decomposed by dimension. $\square$

Crainic and Struchiner in [CS13] proved that

Theorem 4.25. *A proper Lie groupoid $G \rightrightarrows M$ is linearizable.*

As alluded in the introduction, earlier work in this direction was done by Weinstein [Wei02] and Zung [Zun06]. In any case, Theorem 4.25 and Proposition 4.24 give:

Corollary 4.26. *If $G \rightrightarrows M$ is a source-connected linearizable Lie groupoid, $\pi^* : \Omega^\bullet(M/G) \to \Omega_b^\bullet(M, G)$ is an isomorphism. In particular, this is true for proper Lie groupoids.*

Proof. By Theorem 4.25, proper Lie groupoids are linearizable, hence the second assertion. Let $\mathcal{F}_G$ be the singular foliation consisting of the orbits of linearizable G. By Proposition 4.24, $\mathcal{F}_G$ is decomposed by dimension. Hence by Theorem 4.20, π^* is an isomorphism $\Omega^\bullet(M/\mathcal{G}) = \Omega^\bullet(M/\mathcal{F}_\mathcal{G}) \to \Omega_b^\bullet(M, \mathcal{F}_\mathcal{G})$. By source-connectedness of $\mathcal{G}$, and Proposition 4.7, $\Omega_b^\bullet(M, \mathcal{F}_G) = \Omega_b^\bullet(M, G)$, which completes the proof. $\square$

The second assertion in the corollary was also proved by Watts [Wat22], and in fact Watts did not require source-connectedness. However, Watts relied strongly on properness; he invoked compactness of the isotropy groups G_x, and then applied his and Karshon's [KW16] earlier result that π^* is an isomorphism whenever G is the action groupoid of a Lie group action on M with properly acting identity component. This earlier result also relied on compactness of the isotropy groups, in this case of the given Lie group. Our argument does not require properness, although we are still working to adapt the argument for non-source-connected Lie groupoids.

Beyond decomposition by dimension

To finish, we show that pullback by the quotient map $\pi : M \to M/\mathcal{F}$, namely $\pi^* : \Omega^\bullet(M/\mathcal{F}) \to \Omega_b^\bullet(M, \mathcal{F})$, is an isomorphism for a broader class of singular foliations.

Theorem 4.27. *Suppose the singular foliation $(M, \mathcal{F})$ is such that the pullback by $\pi : M_{>0} \to M_{>0}/\mathcal{F}_{>0}$ is an isomorphism $\pi^* : \Omega^\bullet(M_{>0}/\mathcal{F}_{>0}) \to \Omega_b^\bullet(M_{>0}, \mathcal{F}_{>0})$. Then the pullback by the quotient $\pi : M \to M/\mathcal{F}$ is an isomorphism $\pi^* : \Omega^\bullet(M/\mathcal{F}) \to \Omega_b^\bullet(M, \mathcal{F})$.*

Recalling Remark 4.14, note that $M_{>0}$ is an open susbet of M, so the above statement requires no assumption in order to treat $M_{>0}$ as a manifold with singular foliation $\mathcal{F}_{>0}$.

Proof. This proof uses the same ideas as Theorem 4.20. Letting $\alpha \in \Omega^\bullet_b(M, \mathcal{F})$, all we need to show is that α comes from the quotient. We use Proposition 4.4 (i). Let $p, q : \mathsf{U} \to M$ be plots such that $\pi \circ p = \pi \circ q$. Set

$$\begin{aligned} A &:= p^{-1}(M_{>0}) \quad (= q^{-1}(M_{>0})), \text{ which is open in } \mathsf{U} \\ B &:= p^{-1}(M_{=0}) \quad (= q^{-1}(M_{=0})). \end{aligned}$$

As before, to show $\alpha(p) = \alpha(q)$, it suffices to show equality on A and on $\mathrm{int}(B)$. For A, this is a direct result of the assumption and the fact $M_{>0}$ is open, so that $(M_{>0}, \mathcal{F}_{>0})$ is a singular foliation of a manifold. For $r \in \mathrm{int}(B)$, note that $\pi \circ p(r) = \pi \circ q(r)$ reduces to $p(r) = q(r)$, because the 0-leaves are just points. Therefore $p = q$ on $\mathrm{int}(B)$, and $\alpha(p) = \alpha(q)$ on $\mathrm{int}(B)$ follows immediately. □

Here are some singular foliations that fall under the assumptions of Theorem 4.27.

Example 4.28.

- Every singular foliation with leaves of dimension at most 1. In this case, the foliation $(M_{>0}, \mathcal{F}_{>0})$ is simply the regular foliation of $M_{>0} = M_{=1}$ by the 1-leaves, hence the pullback by $\pi : M_{>0} \to M_{>0}/\mathcal{F}_{>0}$ is an isomorphism by Theorem 4.11. This includes every singular foliation of $\mathbb{R}$, and any singular foliation induced by any 1-dimensional Lie group action.

- Every singular foliation that is decomposed by dimension. In this case, the singular foliation $(M_{>0}, \mathcal{F}_{>0})$ is decomposed by dimension, and then Theorem 4.20 says the pullback by $\pi : M_{>0} \to M_{>0}/\mathcal{F}_{>0}$ is an isomorphism. Note the collection of singular foliations decomposed by dimension is strictly contained in the collection of singular foliations for which Theorem 4.27 applies, because for instance if $M = \mathbb{R}$, then $M_{=0}$ can potentially be any closed set.

- Every singular foliation whose singular leaves (leaves L such that every neighbourhood of every point contains leaves of a strictly higher dimension) are only points. In this case $(M_{>0}, \mathcal{F}_{>0})$ is a disjoint union of open, foliated submanifolds, hence decomposed by dimension, and the second bullet applies. For instance, we can take the singular foliation of $\mathbb{R}^2$ consisting of horizontal lines on $y > 0$, and points on $y \leq 0$. This is not decomposed by dimension, but the singular leaves are the points on the x-axis, so Theorem 4.27 applies (note this example also falls under our first bullet).

The first example is particularly interesting, because it suggests that pullback by the quotient is an isomorphism for a very general class of singular foliations. However, at this time we do not have a result analogous to Theorem 4.27 for leaves of dimension $k \geq 1$. Therefore, if a singular foliation $\mathcal{F}$ has leaves of dimension at least 1, and is not stratified by dimension, none of our theorems here readily apply. Locally, we may use Stefan's foliation charts from [Ste74] to straighten the leaves of lowest dimension in a neighbourhood, and then collapse these to return us to a situation with 0-dimensional leaves. This suggests an induction might be possible. But even so, this is only a local procedure, and it is not currently clear how to conclude π^* is an isomorphism from knowing this is true locally. At the time of writing, we are not aware of any examples for which the pullback fails to be an isomorphism.

5 QUASIFOLDS

Quasifolds were introduced by Elisa Prato in [Pra99], as a generalization of manifolds and orbifolds. Whereas manifolds are locally modelled by Cartesian spaces,[1] and orbifolds are locally modelled by quotients of Cartesian spaces by finite group actions, quasifolds are locally modelled by quotients of Cartesian spaces by countable group actions. These spaces often have very coarse topologies. For example, the irrational tori $T_\alpha := \mathbb{R}/(\mathbb{Z} + \alpha\mathbb{Z})$ (for α irrational) have trivial quotient topology. In contrast, when we view irrational tori as diffeological spaces, Donato and Iglesias-Zemmour [DI85] proved that T_α and T_β are diffeomorphic as diffeological spaces if and only if α and β are related by a fractional linear transformation with integer coefficients. *Diffeological quasifolds*, defined by Iglesias-Zemmour and Prato in [IZP21], are diffeological spaces that are, at each point, locally diffeomorphic to a quotient space $\mathbb{R}^n/\Gamma$, for a countable group Γ acting *affinely*[2] on $\mathbb{R}^n$. Special cases include orbifolds [IKZ10], and irrational tori. The groups Γ may change from point to point. As diffeological spaces, diffeological quasifolds inherit a notion of smooth maps and a de Rham complex of differential forms.

A "higher" approach to orbifolds is to define them as Lie groupoids that are, at each point, locally isomorphic to the restriction of an action groupoid $\Gamma \ltimes \mathbb{R}^n$, for a finite group Γ acting linearly on $\mathbb{R}^n$. We similarly define a *quasifold groupoid* to be a Lie groupoid that is, at each point, locally isomorphic to the restriction of the action groupoid $\Gamma \ltimes \mathbb{R}^n$, for a countable group Γ acting affinely on $\mathbb{R}^n$. The groups Γ may change from point to point.

Our main result is that the categories of diffeological quasifolds and quasifold groupoids are equivalent, after restricting to local isomorphisms and effective quasifold groupoids. This completes and extends earlier work about orbifolds by Masrour Zoghi and Yael Karshon [Zog10]. We place quasifold groupoids in the bicategory **Bi**,[3] whose objects are Lie groupoids, whose arrows are principal bibundles, and whose 2-arrows are morphisms of bibundles. We introduce the notion of a *locally invertible* bibundle (Definition 2.30), and the sub-bicategory of effective quasi-

1 $\mathbb{R}^n$ for some n.
2 see Remark 5.11
3 using Lerman's notation in [Ler10]

fold groupoids $\mathbf{QfoldGrpd}^{\text{loc-iso}}_{\text{eff}}$, whose objects are effective quasifold groupoids, whose arrows are locally invertible bibundles, and whose 2-arrows are morphisms of bibundles. We place diffeological quasifolds in the category **Diffeol** of diffeological spaces, whose objects are diffeological spaces, and whose arrows are diffeologically smooth maps. We view this as a bicategory with the identity 2-arrows. We introduce the subbicategory of diffeological quasifolds $\mathbf{DiffeolQfold}^{\text{loc-iso}}$, whose objects are diffeological quasifolds, whose arrows are local diffeomorphisms, and with identity 2-arrows.

There is a quotient functor of bicategories $\mathbf{F} : \mathbf{Bi} \to \mathbf{Diffeol}$, that takes a groupoid to its orbit space. Our main result is Theorem 5.32:

Theorem. *The quotient functor* $\mathbf{F}$ *restricts to a functor of bicategories* $\mathbf{F}_{Quas} : \mathbf{QfoldGrpd}^{loc\text{-}iso}_{eff} \to \mathbf{DiffeolQfold}^{loc\text{-}iso}$ *that is: essentially surjective, surjective on arrows, and injective on arrows up to 2-isomorphism.*

This applies *mutatis mutandis* to orbifold groupoids and diffeological orbifolds.

By identifying isomorphic principal bibundle between groupoids, we form the Hilsum-Skandalis category **HS** of Lie groupoids, whose objects are Lie groupoids, and whose arrows are isomorphism classes of principal bibundles. Restricting to quasifolds, and using the same notation, the theorem above yields

Corollary. *The functor* $\mathbf{F}_{Quas}$ *gives an equivalence of categories between* $\mathbf{QfoldGrpd}^{loc\text{-}iso}_{eff}$ *(viewed in* **HS***) and* $\mathbf{DiffeolQfold}^{loc\text{-}iso}$*.*

In Iglesias-Zemmour and Prato's recent work [IZP21], which builds on [IZL18], the authors define diffeological quasifolds, and associate to each diffeological quasifold a groupoid, and then a C^* algebra, that is unique up to Morita equivalence. Their construction implies that the quotient functor $\mathbf{F}_{\text{Quas}}$ is essentially surjective, and is full on isomorphisms. They do not view their groupoids as Lie groupoids, nor do they introduce the notion of quasifold groupoids.

Prato originally introduced quasifolds through symplectic geometry in [Pra01], in order to generalize the Delzant construction. Recent work involving quasifolds and symplectic geometry includes [BP18], [BP19], [BPZ19], and [LS19]. We also point the reader to Battaglia and Zaffran's work [BZ15], where the authors describe how to realize toric quasifolds as leaf spaces of a foliation. This viewpoint is relevant to the last part of this article. Hoffman [Hof20] works with not-necessarily-effective quasifold groupoids as stacks. Our results can also be written in terms of (effective) stacks, but we leave this for another paper. In particular, we expect our results to extend parts of Cabrera, del Hoyo, and Pujals's investigations of the stack associated to a discrete dynamical system [CdHP20]. For orbifolds as diffeological spaces, we already mentioned [IKZ10] and [IZL18]. A thorough comparison of the categories of orbifolds as diffeological spaces, orbifold groupoids, and orbifolds as Sikorski differential spaces, can be found in [Wat17].

Each section of this chapter addresses a different component of the main theorem. In Section 5.1, we introduce diffeological quasifolds. In Section 5.2, we introduce quasifold groupoids. In the first subsection, we show that the quotient functor $\mathbf{F}_{\mathrm{Quas}}$ is essentially surjective. In the second subsection we show that the quotient functor $\mathbf{F}_{\mathrm{Quas}}$ is surjective on arrows. In the third subsection, we show that the quotient functor $\mathbf{F}_{\mathrm{Quas}}$ is injective on arrows up to 2-isomorphism. In Section 5.4, we describe two effective actions of $\mathbb{Z}$ on $\mathbb{R}$ whose orbits coincide, but whose action groupoids are not Morita equivalent. Namely, they are not related by an invertible bibundle. Thus, our results do not extend to arbitrary countable group actions. Finally, in Section 5.5, we indicate directions for future research.

5.1 DIFFEOLOGICAL QUASIFOLDS

We now define diffeological quasifolds. We use Iglesias-Zemmour and Prato's definition from [IZP21]. It is similar to the diffeological orbifolds introduced by Iglesias-Zemmour, Karshon, and Zadka in [IKZ10].

Definition 5.1 (Diffeological quasifolds)**.** A *diffeological n-quasifold* is a second-countable diffeological space X such that, for each $x \in X$, there is a D-open neighbourhood U of x, a countable subgroup Γ of affine transformations of $\mathbb{R}^n$, an open, Γ-invariant subset $\mathsf{V} \subseteq \mathbb{R}^n$, and a diffeological diffeomorphism (a *chart*) $F : U \to \mathsf{V}/\Gamma$. Here, V/Γ is equipped with its quotient diffeology, which coincides with the subset diffeology induced from $\mathbb{R}^n/\Gamma$, (cf. Lemma 5.16).

We call such V/Γ a *model diffeological quasifold*, and we call a collection $\{F : U \to \mathsf{V}/\Gamma\}$ of diffeomorphisms whose domains U are an open cover of X a *(diffeological quasifold) atlas* for X.

Remark 5.2. To be a diffeological quasifold is a local condition: given a diffeological quasifold X, for every $x \in X$ and open neighbourhood U' of x, there is an open neighbourhood U of x contained in U' such that U is diffeomorphic to a model quasifold V/Γ. Compare to Remark 5.12.

Denoting the quotient by $\pi : \mathbb{R}^n \to \mathbb{R}^n/\Gamma$, no generality is gained by writing our models as $\pi(\mathsf{V})$ for arbitrary open subsets V of $\mathbb{R}^n$. See Lemma 5.16, and compare this to Remark 5.11. Furthermore, no generality is gained if we assume Γ is merely a countable group acting affinely on $\mathbb{R}^n$ (i.e. if we assume the action homomorphism $\Gamma \to \mathrm{Aff}(\mathbb{R}^n)$ is not necessarily injective). This is because $\mathsf{V}/\Gamma = \mathsf{V}/(\Gamma/\ker(\Gamma))$, where $\ker(\Gamma)$ denotes the subgroup of those $\gamma \in \Gamma$ that act as the identity. Compare this to Remark 5.11.

Definition 5.3 (Category of diffeological quasifolds)**.** The category **DiffeolQfold** is the subcategory of **Diffeol** whose objects are diffeological quasifolds and whose morphisms are smooth maps between them. Restricting morphisms to include only local diffeomorphisms, we get $\mathbf{DiffeolQfold}^{\text{loc-iso}}$.

Example 5.4. The *irrational tori* are important examples of quasifolds. For an irrational $\alpha \in \mathbb{R}$, the irrational torus T_α is the diffeological quotient space $\mathbb{R}/(\mathbb{Z} + \alpha\mathbb{Z})$. Here the group $\mathbb{Z} + \alpha\mathbb{Z}$ is countable, and acts affinely on $\mathbb{R}$ by addition, hence T_α is a model diffeological quasifold. Iglesias-Zemmour and Donato in [DI85] prove that $T_\alpha \cong T_\beta$ if and only if α and β are related by a fractional linear transformation with integer coefficients. On the other hand, irrational tori are trivial as topological spaces, and thus any two are homeomorphic. In Example 5.14, we illustrate the groupoid picture.

Example 5.5. If the groups Γ are all finite subgroups of $\mathrm{GL}(\mathbb{R}^n)$, the corresponding quasifold is a *diffeological orbifold* as defined in [IKZ10]. Its D-topology need not be Hausdorff. By Palais' slice theorem [Pal61], the quotient space of a locally proper Lie group action with finite isotropy groups is a diffeological orbifold. More generally, the quotient space of a locally proper Lie groupoid with finite isotropy groups is a diffeological orbifold, cf. Example 5.15.

Remark 5.6. Our model quasifolds, namely diffeological spaces V/Γ for countable groups Γ acting affinely on $\mathsf{V} \subseteq \mathbb{R}^n$, are the same as in [IZP21], but differ from Prato's original models [Pra99]. In [Pra99], Prato models quasifolds by the topological quotient spaces M/Γ, where M is a connected, simply-connected manifold, and Γ is a discrete group acting smoothly, such that the set of points where the action is free is dense, open, and connected. It is not clear whether these models are equivalent. Nevertheless, we expect the theory that we develop to work equally well for Prato's original quasifolds, because the two key lemmas below still hold.

The two key lemmas below concern lifting properties of maps into model quasifolds. The first is a variant of a theorem in Section 3 of [IZP21], which is similar to Lemma 17 from [IKZ10]. Its proof uses the same technique found in Lemma 5.8 of [Miy23a] and Proposition 5.4 of [KW16].

Lemma 5.7. *Suppose Γ is a countable group acting affinely on $\mathbb{R}^n$, and W is a connected open subset of $\mathbb{R}^n$, and $h : \mathsf{W} \to \mathbb{R}^n$ is a C^1 map preserving Γ-orbits. Then for some $\gamma \in \Gamma$, of the form $\gamma \cdot x = A_\gamma x + b_\gamma$,*

$$h(x) = \gamma \cdot x = A_\gamma x + b_\gamma, \quad \textit{for all } x \in \mathsf{W}.$$

Before proving this lemma, we state a convenient corollary. Recall that two smooth functions $f, g : M \to N$ between manifolds have the same *germ* at $x \in M$ if there is a neighbourhood U of x for which $f|_U = g|_U$; we denote by $\operatorname{germ}_x f$ the equivalence class of functions with the same germ at x as f.

Corollary 5.8. *Suppose Γ is a countable group acting affinely on $\mathbb{R}^n$, and W is a (not necessarily connected) open subset of $\mathbb{R}^n$, and $h : \mathsf{W} \to \mathbb{R}^n$ is a C^1 map preserving Γ-orbits. Then for each $x \in W$, there is some $\gamma \in \Gamma$ such that* $\operatorname{germ}_x h = \operatorname{germ}_x \gamma$.

Proof of Lemma 5.7. For each $\gamma \in \Gamma$, set $\Delta_\gamma := \{x \in \mathsf{W} \mid h(x) = \gamma \cdot x\}$, and denote its interior by Δ_γ°. Since h preserves Γ-orbits, $\mathsf{W} = \bigcup_\Gamma \Delta_\gamma$. Furthermore, each Δ_γ is closed in W, being the pre-image of the diagonal in $\mathsf{W} \times \mathsf{W}$ under the continuous map $x \mapsto (h(x), \gamma \cdot x)$. Therefore, as a consequence of the Baire category theorem, $\bigcup_\Gamma \Delta_\gamma^\circ$ is dense in W.

The function h is affine when restricted to each Δ_γ°, so $Dh = A_\gamma$ on this subset. Thus,

$$Dh\left(\bigcup \Delta_\gamma^\circ\right) = \{A_\gamma \mid \gamma \in \Gamma, \text{ and } \Delta_\gamma^\circ \neq \emptyset\}$$

is discrete. But Dh is continuous, and therefore

$$\overline{Dh\left(\bigcup \Delta_\gamma^\circ\right)} = \overline{Dh\left(\overline{\bigcup \Delta_\gamma^\circ}\right)} = \overline{Dh(\mathsf{W})}$$

is discrete. Since W is connected, this set must be a singleton, and $Dh = A_\gamma$ for some $\gamma \in \Gamma$ on all of W. Thus, only the $\Delta^\circ_{(A_\gamma, b_\gamma)}$ are potentially non-empty.

The difference $h - A_\gamma$ is continuous on W, and restricts to b_γ on each $\Delta^\circ_{(A_\gamma, b_\gamma)}$. By the exact same argument as before, we conclude that precisely one $\Delta^\circ_{(A_\gamma, b_\gamma)}$ is non-empty, and the corresponding γ is the desired element of Γ. □

This next lemma appears as a theorem in Section 4 of [IZP21]. It is similar to Lemma 23 in [IKZ10], where they give a global result for finite group actions. See also Lemmas 1.6 and 1.7 in [Pra01].

Lemma 5.9. *Suppose $f : \mathsf{V}/\Gamma \to \mathsf{V}'/\Gamma'$ is a local diffeomorphism between model diffeological quasifolds. For every $\boldsymbol{x} \in \mathsf{V}/\Gamma$ and every $r \in \boldsymbol{x}$ and $r' \in f(\boldsymbol{x})$, there is a transition $\tilde{f}$ between open subsets of V and V' taking r to r' and lifting f.*

Proof. Denote the quotient maps by π and π', respectively. We first show there is a local lift taking r to r'. The map $f\pi : \mathsf{V} \to \mathsf{V}'/\Gamma'$ is diffeologically smooth, and therefore admits a local lift $\widetilde{f\pi}$ about r. Both r' and $\widetilde{f\pi}(r)$

must be in the same orbit (precisely, in $f(\boldsymbol{x})$), therefore there is some $\gamma' \in \Gamma'$ for which $\gamma' \cdot \widetilde{f\pi}(r) = r'$. Then $\gamma' \circ \widetilde{f\pi}$ is the desired local lift of f. This is illustrated in the following diagram (the dashed arrow indicates "locally defined"):

$$\begin{tikzcd}
(\mathsf{V}, r) \arrow[r, dashed, "\widetilde{f\pi}"] \arrow[d, "\pi"] \arrow[dr, "f\pi"] & (\mathsf{V}', \widetilde{f\pi}(r)) \arrow[r, "\gamma'"] \arrow[d, "\pi'"] & (\mathsf{V}', r') \arrow[dl, "\pi'"] \\
(\mathsf{V}/\Gamma, \boldsymbol{x}) \arrow[r, "f"] & (\mathsf{V}'/\Gamma', f(\boldsymbol{x})). &
\end{tikzcd}$$

Now fix a local lift $\tilde{f} : (\mathsf{V}, r) \dashrightarrow (\mathsf{V}', r')$ of f taking r to r'. By assumption on f, there are open neighbourhoods U of $\boldsymbol{x}$ and U' of $\boldsymbol{x}'$, such that the restriction $f : U \to U'$ is a diffeomorphism. By exactly the same procedure as above, we may lift $f^{-1} : U' \to U$ to a locally-defined function $\tilde{s} : (\mathsf{V}', r') \dashrightarrow (\mathsf{V}, r)$.

Consider the composition $\tilde{s}\tilde{f}$, which is defined on some neighbourhood of r. It preserves Γ-orbits, because both $\tilde{f}$ and $\tilde{s}$ are lifts of f and f^{-1}, respectively. Therefore, by Corollary 5.8, there is some $\gamma \in \Gamma$ such that $\operatorname{germ}_r \tilde{s}\tilde{f} = \operatorname{germ}_r \gamma$. In particular, differentiating yields $D\gamma_r = D\tilde{s}_{r'} D\tilde{f}_r$, so $D\tilde{f}_r$ has a left inverse. An exactly similar argument applied to $\tilde{f}\tilde{s}$ shows that $D\tilde{f}_r$ has a right inverse. By the inverse function theorem, we conclude $\tilde{f}$ restricts to a transition taking r to r'. □

5.2 QUASIFOLD GROUPOIDS

Now we introduce quasifold groupoids. Our definition has not previously appeared in the literature.

Definition 5.10. A *n-quasifold groupoid* is a Lie groupoid $G \rightrightarrows G_0$, with Hausdorff arrow space, such that: for each $x \in G_0$, there is an open neighbourhood U of x, a countable group Γ acting affinely on $\mathbb{R}^n$, an open subset V of $\mathbb{R}^n$, and an isomorphism of Lie groupoids $F : G|_U \to (\Gamma \ltimes \mathbb{R}^n)|_{\mathsf{V}}$. We call a collection $\mathcal{A} = \{F : G|_U \to (\Gamma \ltimes \mathbb{R}^n)|_{\mathsf{V}}\}$ of such Lie groupoid isomorphisms such that the U cover G_0 a *(quasifold) atlas* for G. We call a groupoid of the form $(\Gamma \ltimes \mathbb{R}^n)|_{\mathsf{V}}$ a *model quasifold groupoid*.

Remark 5.11. In this definition, we do not assume Γ acts effectively on $\mathbb{R}^n$ (i.e. that the action homomorphism $\Gamma \to \operatorname{Aff}(\mathbb{R}^n)$ is injective), nor do we assume that V is Γ-invariant.

Remark 5.12. Like with diffeological quasifolds, our notion of quasifold groupoid is *local*, meaning that, for a quasifold groupoid G, for each point x, and each neighbourhood U' of x, there is an open neighbourhood U of x

contained in U' such that $G|_U$ is isomorphic to a model quasifold groupoid $(\Gamma \ltimes \mathbb{R}^n)|_V$. We do not know if, in general, these models can always be chosen with V being Γ-invariant. We do know this is the case if all the groups Γ are finite (meaning the quasifold is an orbifold, cf. Example 5.15).

Definition 5.13. The category **QfoldGrpd** is the sub-bicategory of **Bi** whose objects are quasifold groupoids. Taking only the locally invertible bibundles as arrows, we get the bicategory $\mathbf{QfoldGrpd}^{\text{loc-iso}}$.

Example 5.14. In the setting of Example 5.4, the action groupoids $(\mathbb{Z} + \alpha\mathbb{Z}) \ltimes \mathbb{R}$ for irrational tori are quasifold groupoids. Combining our main Theorem 5.32 with Iglesias and Donato's result in [DI85], we recover the fact that two such action groupoids are Morita equivalent if and only if α and β are related by a fractional linear transformation with integer coefficients.

Example 5.15. Taking the groups Γ to be finite subgroups of $\mathrm{GL}(\mathbb{R}^n)$, we call the resulting quasifold groupoid an *orbifold* groupoid. Kozsul's slice theorem [Kos53] implies that an étale Lie groupoid is an orbifold groupoid if and only if it is locally proper.[4] For the argument, see [MM03, Proposition 5.30]. Many authors represent orbifolds by étale, proper Lie groupoids. In contrast, our orbifold groupoids need not be globally proper. The weighted non-singular branched manifolds of [McD19], which come from Kuranishi atlases, provide examples of locally proper Lie groupoids that are not *a priori* proper.

To show the quotient functor $\mathbf{F} : \mathbf{Bi} \to \mathbf{Diffeol}$ restricts to a functor $\mathbf{QfoldGrpd} \to \mathbf{DiffeolQfold}$, we require the following technical lemma.

Lemma 5.16. *Fix a Lie groupoid $G \rightrightarrows G_0$ and an open subset $U \subseteq G_0$. In the following diagram, the bottom inclusion map is a diffeomorphism with its image, $\pi(U)$, where $\pi(U)$ carries the subset diffeology induced from G_0/G.*

$$\begin{tikzcd}
(U, \textit{subset diffeol.}) \arrow[r, hook] \arrow[d, "\pi_U"] & G_0 \arrow[d, "\pi"] \\
(U/G|_U, \textit{quotient diffeol.}) \arrow[r, hook] & (G_0/G, \textit{quotient diffeol.}).
\end{tikzcd}$$

In particular, if Γ is a Lie group acting smoothly on a manifold M, and $\pi : M \to M/\Gamma$ is the quotient map, and V is an open subset of M, then the quotient diffeology on $\pi(V)$ induced from V coincides with the subset diffeology induced from M/Γ.

4 A Lie groupoid is *locally proper* if about every $x \in G_0$, there is a neighbourhood U such that $G|_U$ is proper.

For an open subset $U \subseteq G_0$, the above lemma justifies using the notation $|U|$ to refer unambiguously to the diffeological spaces $U/G|_U = \pi(U)$.

Proof. Fix a map $p : \mathsf{U} \to \pi(U)$. Suppose that p is a plot for the quotient diffeology. Then it locally lifts to smooth maps to U, hence to G_0. So it is a plot of G_0/G, hence of $\pi(U)$ as a subset of G_0/G. Now suppose that p is smooth with respect to the subset diffeology. Then it locally lifts to smooth maps to G_0. Fix $r \in \mathsf{U}$. Let V be an open neighbourhood r in U, and let $q : \mathsf{V} \to G_0$ be a smooth map such that $\pi \circ q = p|_{\mathsf{V}}$. Choose $x \in U$ such that $p(r) = \pi(x)$. Then $\pi(q(r)) = \pi(x)$. Choose an arrow $g : q(r) \to x$.

Because s is a submersion, we may take a section σ of s about $q(r)$ such that $\sigma(x) = g$. Because t is continuous and $t\sigma(x) \in U$, we may restrict the domain of σ so that $t\sigma$ always lands in U. Then $t\sigma q$, on a sufficiently small domain, is the required local lift of p. □

Proposition 5.17. *The orbit space of an n-quasifold groupoid G is a diffeological n-quasifold: if $\mathcal{A} = \{F : G|_U \to (\Gamma \ltimes \mathbb{R}^n)|_{\mathsf{V}}\}$ is an atlas for G, the induced maps $|\mathcal{A}| = \{|F| : |U| \to \mathsf{V}/\Gamma\}$ form an atlas for $|G|$.*

Proof. Fix an n-quasifold groupoid G, and $x \in G_0$. Let $F : G|_U \to (\Gamma \ltimes \mathbb{R}^n)|_{\mathsf{V}}$ be an element of $\mathcal{A}$ such that U contains x. Since F is a Lie groupoid isomorphism, by Proposition 2.32 it descends to a diffeological diffeomorphism $|F| : |U| \to |(\Gamma \ltimes \mathbb{R}^n)|_{\mathsf{V}}|$. By Lemma 5.16, we find $|F|$ is a diffeomorphism from an open subset of X about $[x]$, to an open subset of $\mathbb{R}^n/\Gamma$. Because G_0 is second-countable, so is $|G|$, so by Definition 5.1, we conclude $|G|$ is a diffeological n-quasifold. □

For the rest of this chapter, we deal with effective quasifold groupoids. We denote by $\mathbf{QfoldGrpd}^{\text{loc-iso}}_{\text{eff}}$ the subcategory of $\mathbf{QfoldGrpd}^{\text{loc-iso}}$ consisting of effective quasifold groupoids.

Corollary 5.18. *The quotient functor $\mathbf{F}$ restricts to a functor $\mathbf{F}_{Quas}$ from $\mathbf{QfoldGrpd}^{\textit{loc-iso}}_{\textit{eff}}$ to $\mathbf{DiffeolQfold}^{\textit{loc-iso}}$.*

Proof. This is Proposition 5.17 combined with Proposition 2.32. □

5.3 AN EQUIVALENCE OF CATEGORIES

From diffeological quasifolds to quasifold groupoids

In this section, we show that the quotient functor $\mathbf{F}_{\text{Quas}}$ is essentially surjective. Namely, to a diffeological quasifold X, equipped with an atlas $\mathcal{A}$, we associate an effective quasifold groupoid $\Gamma(\mathcal{A})$ whose quotient $\mathbf{F}(\Gamma(\mathcal{A}))$ is diffeomorphic to X. As the notation suggests, $\Gamma(\mathcal{A})$ will be a germ groupoid of a pseudogroup.

Definition 5.19. Let $\mathcal{A} = \{F_i : U_i \to \mathsf{V}_i/\Gamma_i\}$ be a countable atlas of a diffeological quasifold X. Let

$$\pi : \bigsqcup \mathsf{V}_i \to \bigsqcup \mathsf{V}_i/\Gamma_i, \quad \text{and} \quad F^{-1} : \bigsqcup \mathsf{V}_i/\Gamma_i \to X$$

be the maps induced by π_i and F_i^{-1}. Denote

$$\Pi := F^{-1}\pi.$$

Let $\Psi(\mathcal{A})$ be the pseudogroup of transitions ψ on $\bigsqcup \mathsf{V}_i$ such that $\Pi\psi = \Pi$. Let $\Gamma(\mathcal{A})$ be the germ groupoid associated to the pseudogroup $\Psi(\mathcal{A})$; see Remark 2.36. We have

$$\Gamma(\mathcal{A}) = \bigsqcup_{i,j} \{\operatorname{germ}_x \psi \mid \psi \in \operatorname{Diff}^{\mathcal{A}}_{\mathrm{loc}}(\mathsf{V}_i, \mathsf{V}_j),\ x \in \operatorname{dom}\psi\},$$

where $\operatorname{Diff}^{\mathcal{A}}_{\mathrm{loc}}(\mathsf{V}_i, \mathsf{V}_j)$ is the set of transitions $\psi : \mathsf{V}_i \dashrightarrow \mathsf{V}_j$ in $\Psi(\mathcal{A})$.

Remark 5.20. For each i, the transitions $\psi \in \operatorname{Diff}^{\mathcal{A}}_{\mathrm{loc}}(V_i, V_i)$ are precisely the transitions of V_i which preserve Γ_i-orbits. By Corollary 5.8, for each $x \in \operatorname{dom}\psi$, there is some $\gamma \in \Gamma_i$ such that $\operatorname{germ}_x \psi = \operatorname{germ}_x \gamma$. Therefore $\Gamma(\mathcal{A})|_{\mathsf{V}_i} = \{\operatorname{germ}_x \gamma \mid x \in \mathsf{V}_i,\ \gamma \in \Gamma_i\}$, and $\mathsf{V}_i/(\Gamma(\mathcal{A})|_{\mathsf{V}_i}) = \mathsf{V}_i/\Gamma_i$.

The groupoid $\Gamma(\mathcal{A})$ appears in [IZL18] and [IZP21] as a diffeological groupoid. We show $\Gamma(\mathcal{A})$ is an effective quasifold groupoid; in particular, it is Lie and Hausdorff.

Lemma 5.21. *For a diffeological quasifold X equipped with countable atlas $\mathcal{A}$, the Lie groupoid $\Gamma(\mathcal{A})$ is an effective quasifold groupoid.*

Proof. Since $\Gamma(\mathcal{A})$ is a germ groupoid, it is effective. We show $\Gamma(\mathcal{A})$ has an atlas of quasifold charts. Consider a chart $F : U \to \mathsf{V}/\Gamma$ in $\mathcal{A}$ with $\mathsf{V} \subseteq \mathbb{R}^n$ and Γ is a subgroup of $\operatorname{Aff}(\mathbb{R}^n)$.

By Remark 5.20, elements of $\Gamma(\mathcal{A})|_{\mathsf{V}}$ are precisely the germs of elements of Γ. We may then consider the surjective map

$$\Gamma \times \mathsf{V} \to \Gamma(\mathcal{A})|_{\mathsf{V}}, \quad (\gamma, x) \mapsto \operatorname{germ}_x \gamma. \tag{5.1}$$

This is also injective, because for affine transformations, if $\gamma \neq \gamma'$, then $\operatorname{germ}_x \gamma \neq \operatorname{germ}_x \gamma'$.

For fixed γ, the induced map $\mathsf{V} \to \Gamma(\mathcal{A})|_{\mathsf{V}}$ is simply $\operatorname{germ}\gamma$, which is a smooth diffeomorphism. Since Γ is discrete, we conclude the map in (5.1) is a local diffeomorphism; because it is bijective, it is a diffeomorphism. Together with the identity on the base, the map (5.1) gives the desired functor from the model quasifold.

Now we show $\Gamma(\mathcal{A})$ is Hausdorff. Take any two distinct elements of $\Gamma(\mathcal{A})$. Write them as $\operatorname{germ}_x \psi$ and $\operatorname{germ}_{x'} \psi'$, for $\psi : \mathsf{V}_i \dashrightarrow \mathsf{V}_j$ and $\psi' : \mathsf{V}_{i'} \dashrightarrow \mathsf{V}_{j'}$. If $i \neq i'$ or $j \neq j'$, then ψ and ψ' have disjoint domains or codomains. The corresponding subsets of $\Gamma(\mathcal{A})$ are the desired disjoint neighbourhoods. Now suppose that $i = i'$ and $j = j'$. There are three cases.

Case 1 If $x \neq x'$, separate x and x' by disjoint neighbourhoods U and U'. The required neighbourhoods are the subsets of $\Gamma(\mathcal{A})$ corresponding to $\psi|_{\mathsf{U}}$ and $\psi'|_{\mathsf{U}'}$.

Case 2 If $x = x'$ but $\psi(x) \neq \psi'(x)$, take neighbourhoods U and U' of x such that $\psi(\mathsf{U})$ and $\psi'(\mathsf{U}')$ are disjoint. The required neighbourhoods are the subsets of $\Gamma(\mathcal{A})$ corresponding to $\psi|_{\mathsf{U}}$ and $\psi'|_{\mathsf{U}'}$.

Case 3 If $x = x'$ and $\psi(x) = \psi'(x)$, then $\operatorname{germ}_x(\psi^{-1}\psi')$ is an element of $\Gamma(\mathcal{A})|_{\mathsf{V}_i}$, which by Remark 5.20, we can identify with some $\gamma \in \Gamma_i$. Because $\operatorname{germ}_x \psi \neq \operatorname{germ}_x \psi'$, this γ is not the identity. Therefore $\operatorname{germ}_y \psi \neq \operatorname{germ}_y \psi'$ for all y in some small neighbourhood U of x. The required neighbourhoods are the subsets of $\Gamma(\mathcal{A})$ corresponding to $\psi|_{\mathsf{U}}$ and $\psi'|_{\mathsf{U}}$.

□

Thus we have associated to X the effective quasifold groupoid $\Gamma(\mathcal{A})$. Now we show that the functor $\mathbf{F}$ respects this assignment.

Proposition 5.22. *Fix a diffeological quasifold X with countable atlas $\mathcal{A}$, as in Definition 5.19. In the notation from Definition 5.19, the map Π descends to a diffeological diffeomorphism $|\Pi| : \bigsqcup \mathsf{V}_i / \Gamma(\mathcal{A}) \to X$. In particular, $\mathbf{F}(\Gamma(\mathcal{A})) \cong X$.*

Proof. The orbits of $\Gamma(\mathcal{A})$ are the equivalence classes of the relation: $x \sim \psi(x)$, for some i and j and $\psi \in \operatorname{Diff}^{\mathcal{A}}_{\text{loc}}(\mathsf{V}_i, \mathsf{V}_j)$. Therefore Π descends to the quotient. By results from diffeology, $|\Pi|$ is diffeologically smooth. Because Π is onto, it descends to a surjective map. It remains to show injectivity of $|\Pi|$ and smoothness of its inverse.

- $|\Pi|$ is injective:

  Suppose $\Pi(x) = \Pi(x')$, where $x \in \mathsf{V}_i$ and $x' \in \mathsf{V}_j$. This means $F_i^{-1}\pi_i(x) = F_j^{-1}\pi_j(x')$. Applying F_j on both sides, and denoting $F_{ij} := F_j F_i^{-1}$, we get
  $$F_{ij}\pi_i(x) = \pi_j(x').$$
  The map F_{ij} is a transition $\mathsf{V}_i/\Gamma_i \dashrightarrow \mathsf{V}_j/\Gamma_j$. By Lemma 5.9, we can lift it to a transition $\psi_{ij} : \mathsf{V}_i \dashrightarrow \mathsf{V}_j$ sending x to x'. Therefore x and x' are in the same orbit, hence the induced map $|\Pi|$ is injective.

- $|\Pi|$ has an inverse:

  The inclusion $\iota : \mathsf{V}_i \hookrightarrow \bigsqcup \mathsf{V}_i$ is smooth, and if x and x' are in the same Γ_i orbit, then $\iota(x)$ and $\iota(x')$ are in the same $\Gamma(\mathcal{A})$ orbit. Therefore the inclusion descends to a smooth map $\bar{\iota} : \mathsf{V}_i/\Gamma_i \to \bigsqcup \mathsf{V}_i/\Gamma(\mathcal{A})$. The composition $\iota F_i : U_i \to \iota(\mathsf{V}_i/\Gamma_i)$ is smooth, and it is the inverse of the restriction $|\Pi| : \iota(\mathsf{V}_i/\Gamma_i) \to U_i$. This shows $|\Pi|$ is a local diffeomorphism, and since it is bijective, it is a diffeomorphism.

$\square$

Lifting local diffeomorphisms between diffeological quasifolds

In this section we show that the quotient functor $\mathbf{F}_{\mathrm{Quas}}$ is surjective on arrows. In other words, given effective quasifold groupoids G and H, and a local diffeomorphism $f : |G| \to |H|$, we construct a locally invertible bibundle $P : G \to H$ such that $|P| = f$. First, we replace the effective quasifolds G and H with convenient germ groupoids.

Definition 5.23. For an étale Lie groupoid G, define the pseudogroup $\mathrm{Diff}^G_{\mathrm{loc}}(G_0)$ to consist of all transitions ψ of G_0 that preserve G-orbits. Denote its germ groupoid by

$$\Gamma^G := \Gamma(\mathrm{Diff}^G_{\mathrm{loc}}(G_0)).$$

Note that the pseudogroup $\Psi(G)$ of local bisections of G (see Example 2.35) is a subset of $\mathrm{Diff}^G_{\mathrm{loc}}(G_0)$.

Lemma 5.24. *If G is a quasifold groupoid, then $\Psi(G) = \mathrm{Diff}^G_{loc}(G_0)$, and hence $|G| = |\Gamma^G|$.*

Proof. Fix $\psi \in \mathrm{Diff}^G_{\mathrm{loc}}(G_0)$. It suffices to show that about every $x \in \operatorname{dom}\psi$, there is a neighbourhood on which ψ restricts to an element of $\mathrm{Diff}^G_{\mathrm{loc}}(G_0)$.

Case 1: $\psi(x) = x$. Choose a quasifold groupoid chart $F : G|_U \to (\Gamma \ltimes \mathbb{R}^n)|_{\mathsf{V}}$ about x. The (partially defined) map $F_0 \psi F_0^{-1}$ restricts to a transition $V \dashrightarrow V$ about $F_0(x)$ that preserves Γ-orbits. By Corollary 5.8, we may choose $\gamma \in \Gamma$ such that $\mathrm{germ}_{F_0(x)} F_0 \psi F_0^{-1} = \mathrm{germ}_{F_0(x)} \gamma$. Consider the partially defined map

$$\sigma : U \dashrightarrow \mathsf{G}|_U, \quad x' \mapsto F^{-1}(\gamma, F_0(x')).$$

This is a section of s, and composing with t yields,

$$tF^{-1}(\gamma, F_0(x')) = F_0^{-1}(\gamma \cdot F_0(x')) = \psi(x').$$

Therefore, near x, we have $\psi = t\sigma \in \Psi(G)$.

Case 2: $\psi(x) \neq x$. Since preserves G-orbits, there exists an arrow $g \in G$ such that $g : x \to \psi(x)$. Let σ be a section of s through g. Then $(t\sigma)^{-1}\psi$ is an element of $\mathrm{Diff}^G_{\mathrm{loc}}(G_0)$ fixing x. By Case 1, this is in $\Psi(G)$. Therefore, near x, we have $\psi = (t\sigma)((t\sigma)^{-1}\psi) \in \Psi(G)$.

□

Corollary 5.25. *By the correspondence* (2.4)*, we see* $\mathrm{Eff}(G) = \Gamma^G$. *If G is also effective, then the effect functor is an isomorphism* $G \cong \mathrm{Eff}(G) = \Gamma^G$.

We are almost in the position to show that the quotient functor $\mathbf{F}_{\mathrm{Quas}}$ is surjective on arrows. We require one more lemma.

Lemma 5.26. *Suppose G and H are effective quasifold groupoids. Assume $f : |G| \to |H|$ is a local diffeomorphism. Denote the quotient maps by*

$$\pi : G_0 \to |G| \quad \text{and} \quad \pi' : H_0 \to |H|.$$

Then for every $x \in G_0$, and $x' \in H_0$, such that $f(\pi(x)) = \pi'(x')$, there is a transition φ taking x to x' that lifts f.

Proof. Take x and x', and fix quasifold groupoid charts

$$F : G|_U \to (\Gamma \ltimes \mathbb{R}^n)|_V, \quad F' : H|_{U'} \to (\Gamma' \ltimes \mathbb{R}^n)|_{V'}$$

about x and x'. The partially defined map

$$|F'| \circ f \circ |F^{-1}| : V/\Gamma \dashrightarrow V'/\Gamma'$$

is a local diffeomorphism taking $|F^{-1}|(\pi(x))$ to $|F'|(\pi'(x'))$. Therefore, by Lemma 5.9, this lifts to a transition $V \dashrightarrow V'$ taking $F_0(x)$ to $F'_0(x')$. Precomposing this transition with F_0, and post-composing it with F'_0, yields the required lift φ of f taking x to x'. □

Proposition 5.27. *Suppose G and H are effective quasifold groupoids. Assume $f : |G| \to |H|$ is a local diffeomorphism. Then there is a locally invertible bibundle $P : G \to H$ such that $|P| = f$. If f is a diffeomorphism, then P is invertible.*

Proof. Let

$$\pi : G_0 \to |G| \quad \text{and} \quad \pi' : H_0 \to |H|$$

be the quotient maps.

By Corollary 5.25, the effect functor is an isomorphism $G \cong \Gamma^G$, where Γ^G is the germ groupoid of the pseudogroup consisting of transitions on G_0 preserving G-orbits (Definition 5.23). We similarly have $H \cong \Gamma^H$. Since the effect functor descends to the identity on the orbit space, it suffices to give a locally invertible bibundle $P : \Gamma^G \to \Gamma^H$ lifting f.

Set

$$Q := \{\operatorname{germ}_x \varphi \mid \varphi \text{ is a transition } G_0 \dashrightarrow H_0 \text{ lifting } f\},$$

with its manifold structure given by the charts $\operatorname{germ}_x \varphi \mapsto x$ (cf. Remark 2.36). We introduce a left action of Γ^H on Q along the anchor $a' : \operatorname{germ}_x \varphi \mapsto \varphi(x)$ by

$$\operatorname{germ}_{\varphi(x)} \psi' \cdot \operatorname{germ}_x \varphi := \operatorname{germ}_x \psi' \varphi.$$

We introduce a right action of Γ^G on Q along the anchor $a : \operatorname{germ}_x \varphi \mapsto x$ by

$$\operatorname{germ}_x \varphi \cdot \operatorname{germ}_{\psi^{-1}(x)} \psi := \operatorname{germ}_{\psi^{-1}(x)} \varphi\psi.$$

Both actions are smooth because multiplication of germs is smooth. In a diagram, we have

$$\begin{tikzcd} \Gamma^H \circlearrowright & Q \arrow[dl, "a'"'] \arrow[dr, "a"] & \circlearrowleft \Gamma^G \\ H_0 & & G_0. \end{tikzcd}$$

Both a and a' are submersions (in fact, they are local diffeomorphisms). The anchor a' is Γ^G-invariant, and a is Γ^H-invariant, and the actions commute. Therefore $Q : \Gamma^H \to \Gamma^G$ is a bibundle. We will prove that

- $Q \xrightarrow{a} G_0$ is left Γ^H-principal;
- For every $x_0 \in G_0$ and $y_0 \in a'(a^{-1}(x_0))$, there are neighbourhoods U of x_0 and V of y_0 such that $Q|_V^U : \Gamma^H|_V \to \Gamma^G|_U$ is invertible.

It immediately follows from these claims that swapping the actions of Γ^G and Γ^H, as in Example 2.21, yields a locally invertible bibundle $P : \Gamma^G \to \Gamma^H$, which still has total space Q. Because the elements of Q are germs of lifts of f, the bibundle P descends to f.

We prove the first item. Let $x_0 \in G_0$, and fix a transition $\varphi_0 : G_0 \dashrightarrow H_0$ defined near x_0 lifting f. Denote the domain of φ_0 by U and set $U' := \varphi_0(U)$. Assume U is small enough so that the restriction $f : |U| \to |U'|$ is a diffeomorphism. The map σ defined by

$$\sigma : U \to Q, \quad x \mapsto \operatorname{germ}_x \varphi_0$$

is a smooth section of s. Now set

$$\Phi : a^{-1}(U) \to \Gamma^H {}_s\times_{a'\sigma} U, \quad \operatorname{germ}_x \varphi \mapsto (\operatorname{germ}_{\varphi_0(x)} \varphi\varphi_0^{-1}, x). \tag{5.2}$$

This is well-defined because $\varphi\varphi_0^{-1}$ is a transition on H_0 preserving H-orbits: by choice of U, we have the commutative diagram

$$\begin{tikzcd}
U \arrow[r, bend left, "\varphi_0"] \arrow[d, "\pi_a"'] \arrow[rr, dashed, bend left=40, "\varphi"] & U' \arrow[l, bend left, "\varphi_0^{-1}"] \arrow[d, "\pi'_b"] & \varphi(U) \arrow[dl, "\pi'_c"] \\
{|U|} \arrow[r, bend left, "f"] & {|U'|,} \arrow[l, bend left, "f^{-1}"]
\end{tikzcd}$$

(where we use the subscripts on the quotient maps to distinguish them), and therefore

$$\pi'_c\varphi\varphi_0^{-1} = f\pi_a\varphi_0^{-1} = ff^{-1}\pi'_b = \pi'_b.$$

The map Φ has an inverse given by

$$\Gamma^H {}_s\times_{a'\sigma} U \to a^{-1}(U), \quad (\mathrm{germ}_{\varphi_0(x)} \psi', x) \mapsto \mathrm{germ}_x \psi'\varphi_0.$$

This is well-defined because ψ' preserves H-orbits, so $\psi'\varphi_0$ remains a transition lifting f. Both Φ and its inverse are smooth because locally only the base-points of the germs vary (smoothly) as x moves through U. Noting that Φ is Γ^H-equivariant, and fits into the diagram

$$\begin{tikzcd}
\Gamma^H {}_s\times_{a'\sigma} U \arrow[dr, "\mathrm{pr}_2"] & a^{-1}(U) \arrow[l, "\Phi"'] \arrow[d, "a"] \\
& U,
\end{tikzcd}$$

we conclude that $Q \xrightarrow{a} G_0$ is left Γ^H-principal.

We now prove the second item. Choose $x_0 \in G_0$, and $y_0 \in a'(a^{-1}(x_0))$. Using Lemma 5.26, fix a transition $\varphi_0 : G_0 \dashrightarrow H_0$ defined near x_0 lifting f, so that $y_0 = a'(\mathrm{germ}_{x_0} \varphi_0)$. Reusing the notation above, denote the domain of φ_0 by U, and set $U' := \varphi_0(U)$. Assume U is small enough so that the restriction $f : |U| \to |U'|$ is a diffeomorphism. We claim

$$\begin{tikzcd}
\Gamma^H|_{U'} \circlearrowright & Q|^U_{U'} \arrow[dl, "a'"'] \arrow[dr, "a"] & \circlearrowleft \Gamma^G|_U \\
U' & & U.
\end{tikzcd}$$

is an invertible bibundle (in fact, we show that the bundles on both sides are globally trivializable).

To show $Q|_{U'}^{U} \xrightarrow{a} U$ is left $\Gamma^H|_{U'}$-principal, define a section σ of s by

$$\sigma : U \to Q|_{U'}^{U}, \quad x \mapsto \operatorname{germ}_x \varphi_0,$$

and set

$$\Phi : Q|_{U'}^{U} \to \Gamma^H|_{U'} \,{}_{s}\!\times_{a'\sigma} U, \quad \operatorname{germ}_x \varphi \mapsto (\operatorname{germ}_{\varphi_0(x)} \varphi\varphi_0^{-1}, x).$$

The section σ and map Φ trivialize the bundle in the same way they gave a local trivialization for $Q \xrightarrow{a} G_0$, and for brevity we will not repeat the details here.

To show $Q|_{U'}^{U} \xrightarrow{a'} U'$ is right $\Gamma^G|_U$-principal, define a section σ of a' by

$$\sigma : U' \to Q|_{U'}^{U} \quad y \mapsto \operatorname{germ}_{\varphi_0^{-1}(y)} \varphi_0,$$

and set

$$\Phi : Q|_{U'}^{U} \to U' \,{}_{a\sigma}\!\times_t \Gamma^G|_U, \quad \operatorname{germ}_x \varphi \mapsto (\varphi(y), \operatorname{germ}_y \varphi_0^{-1}\varphi).$$

The map Φ is well-defined, because $\varphi_0^{-1}\varphi$ preserves G-orbits: by choice of U, we have the commutative diagram

$$\begin{tikzcd}
\operatorname{dom}\varphi \arrow[r, hook] \arrow[d, "\pi_a"] \arrow[rr, dashed, bend left, "\varphi"] & U \arrow[r, bend left, "\varphi_0"] \arrow[d, "\pi_b"'] & U' \arrow[l, bend left, "\varphi_0^{-1}"] \arrow[d, "\pi_c'"] \\
|\operatorname{dom}\varphi| \arrow[r, hook] & |U| \arrow[r, bend left, "f"] & |U'|, \arrow[l, bend left, "f^{-1}"]
\end{tikzcd}$$

so on $\operatorname{dom}\varphi \subseteq U$,

$$\pi_b\varphi_0^{-1}\varphi = f^{-1}\pi_c'\varphi = f^{-1}f\pi_a = \pi_a.$$

The inclusion $\operatorname{dom}\varphi \subseteq U$ follows from the assumption that $\operatorname{germ}_x \varphi \in Q|_{U'}^{U}$. Note that if we were trying to show $Q \xrightarrow{a'} H_0$ is right Γ^G-principal (and hence Q is invertible), this is the step where we could fail. In this setting, the domain of φ need not be inside of U, and so $f^{-1}f$ need not fix elements of $|\operatorname{dom}\varphi|$. On the other hand, if f is a diffeomorphism, then

$f^{-1}f$ is always the identity, and the proof would proceed without issue. The inverse of Φ is given by

$$U' \; {}_{a\sigma}\!\times_t \Gamma^G|_U \to Q|^U_{U'}, \quad (x', \operatorname{germ}_{\psi^{-1}(\varphi_0^{-1}(x'))} \psi) \mapsto \operatorname{germ}_{\psi^{-1}(\varphi_0^{-1}(x'))} \varphi_0\psi.$$

This is well-defined because ψ preserves G-orbits, so $\varphi_0\psi$ remains a transition lifting f. Both Φ and its inverse are smooth because locally only the base-points of the germs vary (smoothly) as y moves through U'. Finally, Φ is $\Gamma^G|_U$-equivariant, and fits into the diagram

$$\begin{tikzcd} G|^U_{U'} \arrow[r, "\Phi"] \arrow[d, "a'"] & U' \; {}_{a\sigma}\!\times_t \Gamma^G|_U \arrow[dl, "\mathrm{pr}_1"] \\ U', & \end{tikzcd}$$

so we conclude that $Q|^U_{U'} \xrightarrow{a'} U'$ is right $\Gamma^G|_U$-principal. □

We will exhibit this proposition in two examples: the case of a diffeomorphism between model quasifolds, and more specifically diffeomorphisms between irrational tori.

Example 5.28. Suppose that $G := \Gamma \ltimes \mathbb{R}^n$ and $H := \Gamma' \ltimes \mathbb{R}^n$ are model quasifold groupoids, and that $f : \mathbb{R}^n/\Gamma \to \mathbb{R}^n/\Gamma'$ is a diffeomorphism. From the proof of Proposition 5.27,

$$P := \{\operatorname{germ}_x \varphi \mid \varphi \text{ is a transition on } \mathbb{R}^n \text{ lifting } f\}$$

is an invertible bibundle between the action groupoids G and H. By Example 2.24, P is simultaneously a Γ and Γ'-principal bundle. The Γ and Γ' actions on P are

$$\gamma \cdot \operatorname{germ}_x \varphi := \operatorname{germ}_x \varphi \circ \gamma^{-1} \text{ and } (\operatorname{germ}_x \varphi) \cdot \gamma' := \operatorname{germ}_x (\gamma')^{-1} \circ \varphi,$$

and the bundle projections are $a : \operatorname{germ}_x \varphi \mapsto x$ and $a' : \operatorname{germ}_x \varphi \mapsto \varphi(x)$, respectively. Because $P \xrightarrow{a} \mathbb{R}^n$ is a principal right Γ'-bundle, and $\mathbb{R}^n$ is simply-connected, a admits a global (trivializing) section. By definition of the manifold structure on P, a global section of a is necessarily of the form $x \mapsto \operatorname{germ}_x \varphi_0$, where φ_0 is a diffeomorphism of $\mathbb{R}^n$ lifting f. Then $y \mapsto \operatorname{germ}_{\varphi_0^{-1}(y)} \varphi_0$ is a global section of a'.

Under the trivializations induced by these sections, we find that the map

$$\lambda : \Gamma \to \Gamma', \quad \gamma \mapsto \varphi_0 \circ \gamma \circ \varphi_0^{-1} \tag{5.3}$$

(where we interpret $\Gamma \subseteq \mathrm{Aff}(\mathbb{R}^n)$ to define the composition) is a well-defined group isomorphism, and the diffeomorphism $\varphi_0 : \mathbb{R}^n \to \mathbb{R}^n$ intertwines the actions with respect to λ. In other words,

$$(\lambda, \varphi_0) : G \to H$$

is a Lie groupoid isomorphism. Therefore, the following are equivalent:

- $\mathbb{R}^n/\Gamma$ and $\mathbb{R}^n/\Gamma'$ are isomorphic;
- G and H are Morita equivalent;
- G and H are isomorphic, and some isomorphism has the form (λ, φ_0) as above.

We note that [CdHP20, Lemma 3.2] shows that any smooth functor $\Gamma \ltimes \mathbb{R}^n \to \Gamma' \ltimes \mathbb{R}^n$ is of the form (λ, φ), where $\lambda : \Gamma \to \Gamma'$ is a group homomorphism and φ is a smooth λ-equivariant map.

Example 5.29. Let us now specialize the previous example to the case of irrational tori (cf. Examples 5.4 and 5.14). Thus $\Gamma := \mathbb{Z} + \alpha\mathbb{Z}$, and $\Gamma' := \mathbb{Z} + \beta\mathbb{Z}$, and they act on $\mathbb{R}$ by translation. By the previous example, a diffeomorphism $f : T_\alpha \to T_\beta$ lifts to a diffeomorphism $\varphi_0 : \mathbb{R} \to \mathbb{R}$, and this diffeomorphism is equivariant with respect to the group isomorphism λ defined by (5.3). Any isomorphism $\Gamma \to \Gamma'$ has the form $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}(2;\mathbb{Z})$, where

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} (m + \alpha n) := am + bn + \beta(cm + dn).$$

By equivariance of φ_0, we have that for any $m + \alpha n$,

$$\begin{aligned} \varphi_0(m + \alpha n) &= \varphi_0(0) + am + bn + \beta(cm + dn) \\ &= \varphi_0(0) + (a + \beta c)\left(m + \frac{b + \beta d}{a + \beta c} n\right). \end{aligned}$$

In particular, if we choose $m_i + \alpha n_i \to 0$, then by continuity

$$\varphi_0(0) = \lim_{i\to\infty} \varphi_0(m_i + \alpha n_i) = \varphi_0(0) + (a + \beta c) \lim_{i\to\infty} \left(m_i + \frac{b + \beta d}{a + \beta c} n_i\right),$$

and the limit on the right vanishes if and only if

$$\frac{b + \beta d}{a + \beta c} = \alpha. \tag{5.4}$$

Conversely, if (5.4) holds for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}(2;\mathbb{Z})$, then the map $\varphi_0(x) := (a+\beta c)x$ descends to a diffeomorphism $|\varphi| : T_\alpha \to T_\beta$.

We conclude that α and β are conjugate modulo $\mathrm{GL}(2;\mathbb{Z})$ if and only if any of the following equivalent conditions hold:

- T_α and T_β are diffeomorphic;
- $(\mathbb{Z}+\alpha\mathbb{Z}) \ltimes \mathbb{R}$ and $(\mathbb{Z}+\beta\mathbb{Z}) \ltimes \mathbb{R}$ are Morita equivalent;
- $(\mathbb{Z}+\alpha\mathbb{Z}) \ltimes \mathbb{R}$ and $(\mathbb{Z}+\beta\mathbb{Z}) \ltimes \mathbb{R}$ are isomorphic along an isomorphism of the form (λ, φ_0).

The first equivalence reproduces Donato and Iglesias-Zemmour's result in [DI85], the second is our Proposition 5.27, and the last can also be found, in a slightly different form, in Cabrera, del Hoyo, and Pujal's work on discrete dynamics [CdHP20, Proposition 9.1]

Local diffeomorphisms have essentially unique lifts

In this section, we show that the quotient functor $\mathbf{F}_{\mathrm{Quas}}$ is injective on arrows, up to 2-isomorphism. In other words, for two locally invertible bibundles $P, Q : G \to H$ between effective quasifold groupoids that descend to the same map $|P| = |Q| : |G| \to |H|$, we give an isomorphism of bibundles $P \cong Q$. We begin with the case of functors.

Lemma 5.30. *Suppose G, H are effective quasifold groupoids, and $F, K : G \to H$ are smooth functors that induce the same map $|F| = |K| : |G| \to |H|$. If $|F|$ (hence $|K|$) is a local diffeomorphism, then there is a smooth natural transformation $F \to K$.*

Proof. Being effective quasifold groupoids, by Corollary 5.25 the effect functor Eff gives an isomorphism $G \cong \mathrm{Eff}(G) = \Gamma^G$ and $H \cong \mathrm{Eff}(H) = \Gamma^H$ (see Definition 5.23 for Γ^G and Γ^H). Then, to find a smooth natural transformation $F \implies K$, it suffices to find a smooth natural transformation $\mathrm{Eff}^{-1} F\, \mathrm{Eff} \implies \mathrm{Eff}^{-1} K\, \mathrm{Eff}$. So without loss of generality, assume that $F, K : \Gamma^G \to \Gamma^H$ are two smooth functors that induce the same map $|G| \to |H|$.

Because $F, K : G_0 \to H_0$ both lift the local diffeomorphism $|F| = |K|$, they are also local diffeomorphisms; this is a consequence of Corllary 5.8 and Lemma 5.9. Define

$$\alpha : G_0 \to \Gamma^H, \quad \alpha(x) := \mathrm{germ}_{F(x)} K\sigma,$$

where

$$\sigma \text{ is a local inverse of } F \text{ sending } F(x) \text{ to } x. \tag{5.5}$$

Because F is a local diffeomorphism, the germ of σ is uniquely defined by condition (5.5), so α is well-defined. Fixing x and its attendant σ, the map $K\sigma$ is a smooth map defined on a neighbourhood of $F(x)$, and it preserves Γ^H-orbits. Indeed, denoting the quotient maps for G and H by π and π', respectively,

$$\pi' K\sigma = \pi' F\sigma = \pi' \text{ near } x,$$

where $\pi' K = \pi' F$ by the assumption that F and K descend to the same map on the quotient. Thus $\alpha(x) \in \Gamma^H$. We claim that α is a smooth natural transformation $F \implies K$.

To see α is smooth, note that if σ satisfies (5.5) at x_0, then it also satisfies (5.5) for all x near x_0, so for such x we have $\alpha(x) = \operatorname{germ}_{F(x)} K\sigma$ with the same σ. For α to be a natural transformation, we require that $\alpha(x_0)$ is an arrow from $F(x)$ to $G(x)$, and that the following commutes for all arrows $\operatorname{germ}_x \psi \in \Gamma^G$:

$$\begin{tikzcd} F(x) \arrow[r, "\alpha(x)"] \arrow[d, "F(\operatorname{germ}_x \psi)"'] & K(x) \arrow[d, "K(\operatorname{germ}_x \psi)"] \\ F(\phi(x)) \arrow[r, "\alpha(\psi(x))"] & K(\psi(x)). \end{tikzcd} \tag{5.6}$$

By the definition of $\alpha(x)$, its source is $F(x)$, and its target is $K\sigma F(x)$, which is $K(x)$ by condition (5.5). As for the diagram (5.6), fix $\operatorname{germ}_x \psi$. We use the fact that there is some $\psi'_F \in \operatorname{Diff}^H_{\text{loc}}(H_0)$ such that for all $\tilde{x}$ near x,

$$F(\operatorname{germ}_{\tilde{x}} \psi) = \operatorname{germ}_{F(\tilde{x})} \psi'_F.$$

This is a consequence of the assumption that F is a smooth functor, and that each open subset of Γ^H is of the form $\{\operatorname{germ}_{\tilde{x}} \psi' \mid \psi' \in \operatorname{Diff}^H_{\text{loc}}(H_0)\}$. Note that it follows that $F\psi = \psi'_F F$ near x. We similarly have ψ'_K, with $K\psi = \psi'_K K$ near x.

Then

$$\begin{aligned} K(\operatorname{germ}_x \psi) \cdot \alpha(x) &= \operatorname{germ}_{K(x)} \psi'_K \cdot \operatorname{germ}_{F(x)} K\sigma \\ &= \operatorname{germ}_{F(x)} \psi'_K K\sigma \\ &= \operatorname{germ}_{F(x)} K\psi\sigma, \end{aligned}$$

and for the local inverse σ' of F taking $F(\psi(x))$ to $\psi(x)$,

$$\begin{aligned} \alpha(\psi(x)) \cdot F(\operatorname{germ}_x \psi) &= \operatorname{germ}_{F(\psi(x))} K\sigma' \cdot \operatorname{germ}_{F(x)} \psi'_F \\ &= \operatorname{germ}_{F(x)} K\sigma' \psi'_F. \end{aligned}$$

Now, near x, we have $K\psi\sigma F = K\psi$, because σF is the identity near x. Similarly, near x,

$$K\sigma'\psi'_F F = K\sigma' F\psi = K\psi,$$

because $\sigma' F$ is the identity near $\psi(x)$. We therefore have

$$\operatorname{germ}_x K\psi\sigma F = \operatorname{germ}_x K\psi, = \operatorname{germ}_x K\sigma'\psi'_F F,$$

and because F is a local diffeomorphism, we conclude $\operatorname{germ}_{F(x)} K\psi\sigma = \operatorname{germ}_{F(x)} K\sigma'\psi'_F$. Thus (5.6) commutes, and the proof is complete. $\square$

Now we handle the general case.

Proposition 5.31. *Suppose $P, Q : G \to H$ are principal bibundles between effective quasifold groupoids that descend to the same map $|P| = |Q| : |G| \to |H|$, and that $|P|$ is a local diffeomorphism. Then P and Q are isomorphic bibundles.*

Proof. Choose an open cover $\{U_i\}$ of G_0 such that for each i, the right anchors of P and Q admit sections defined on U_i. This is possible because the anchor maps are submersions. Denoting the inclusion $\iota : \bigsqcup U_i \to G_0$, by [Ler10, Lemma 3.37] there are functors $F, K : \iota^* G \to H$ such that

$$\begin{aligned} P \circ \langle \iota \rangle &\text{ is isomorphic to } \langle F \rangle, \text{ and} \\ Q \circ \langle \iota \rangle &\text{ is isomorphic to } \langle K \rangle, \end{aligned}$$

where $\langle \iota \rangle$, $\langle F \rangle$, and $\langle K \rangle$ are the corresponding principal bibundles, as in Example 2.27. Since $\langle \iota \rangle$ is invertible, it suffices to show $\langle F \rangle \cong \langle K \rangle$. But $\iota^* G$ is also an effective quasifold groupoid, and F, K are two smooth functors that induce the same map of quotient spaces, namely $|P| \circ |\langle \iota \rangle| = |Q| \circ |\langle \iota \rangle|$. By Lemma 5.30 and Example 2.27, we find $\langle F \rangle \cong \langle K \rangle$. $\square$

This proposition proves that the quotient functor $\mathbf{F}_{\mathrm{Quas}}$ on effective quasifold groupoids is injective on arrows, up to 2-isomorphism. We gather our results in the main theorem.

Theorem 5.32. *Let $\mathbf{QfoldGrpd}_{\mathit{eff}}^{\mathit{loc\text{-}iso}}$ be the bicategory whose objects are effective quasifold groupoids, whose arrows are locally invertible bibundles, and whose 2-arrows are morphisms of bibundles. Let $\mathbf{DiffeolQfold}^{\mathit{loc\text{-}iso}}$ be the bicategory whose objects are diffeological quasifolds, whose arrows are local diffeomorphisms, and whose 2-arrows are trivial. Then the quotient functor $\mathbf{F}_{\mathit{Quas}} : \mathbf{QfoldGrpd}_{\mathit{eff}}^{\mathit{loc\text{-}iso}} \to \mathbf{DiffeolQfold}^{\mathit{loc\text{-}iso}}$, which is well defined by Corollary 5.18, is essentially surjective, is surjective on arrows, and injective on arrows up to 2-isomorphism.*

Proof.

- It is essentially surjective: given a diffeological quasifold X, take an atlas $\mathcal{A}$. Then $\Gamma(\mathcal{A})$ is an effective quasifold groupoid with orbit space diffeomorphic to X by Proposition 5.22.
- It is surjective on arrows: given a local diffeomorphism $f : |G| \to |H|$, where G and H are effective quasifold groupoids, we get a locally invertible bibundle $P : G \to H$ such that $|P| = f$ from Proposition 5.27.
- It injective on arrows up to 2-isomorphism: if $P, Q : G \to H$ are locally invertible bibundles between effective quasifold groupoids that induce the same map on the orbit spaces, then $P \cong Q$ by Proposition 5.31.

□

Moving to the Hilsum-Skandalis category **HS** (see Remark 2.26) lets us phrase this result more succinctly.

Corollary 5.33. *The quotient functor, from the category whose objects are effective quasifold groupoids, and whose arrows are isomorphism classes of locally invertible bibundles, to the category whose objects are diffeological quasifolds, and whose arrows are local diffeomorphisms, is an equivalence of categories.*

Denote by $\mathbf{QfoldGrpd}_{\text{eff}}^{[\text{loc-iso}]}$ the subcategory of **HS** induced by $\mathbf{QfoldGrpd}_{\text{eff}}^{\text{loc-iso}}$. Corollary 5.33 resolves the question of gluing morphisms in $\mathbf{QfoldGrpd}_{\text{eff}}^{[\text{loc-iso}]}$. This is a question because in **HS**, morphisms generally do not glue, i.e. if P and Q are principal bibundles between Lie groupoids G and H, and there is an open cover $\{U_i\}$ of G_0 such that $P|_{U_i} \cong Q|_{U_i}$ for all i, it does not follow that $P \cong Q$. For examples, see [Ler10, Lemma 3.41] or [HM04, page 2491]. However, for quasifolds:

Corollary 5.34. *If P and Q are locally invertible bibundles between effective quasifold groupoids G and H, and there is an open cover $\{U_i\}$ of G_0 such that $P|_{U_i} \cong Q|_{U_i}$ for all i, then $P \cong Q$.*

Proof. For each i, denote the inclusion $U_i \to G_0$ by ι_{U_i}. By Lemma 2.29 and Proposition 2.32,

$$|P|_{U_i}| = |P \circ \langle \iota_{U_i} \rangle| = |P| \circ |\langle \iota_{U_i} \rangle| = |P||_{|U_i|}.$$

The assumption $P|_{U_i} \cong Q|_{U_i}$ implies that $|P||_{|U_i|} = |Q||_{|U_i|}$. Since the $|U_i|$ cover $|G_0|$, and since diffeologically smooth maps are local, this means $|P| = |Q|$. By Corollary 5.33, we conclude that $P \cong Q$. □

5.4 A NON-EXAMPLE

By Theorem 5.32, given two quasifold groupoids G and H, a diffeomorphism $|G| \to |H|$ lifts to a Morita equivalence of G and H. In this last section, we describe a non-example where this lifting property fails. Fix a smooth function $h : \mathbb{R} \to \mathbb{R}$ that is flat at 0 and positive everywhere else, and such that the vector field

$$\xi := h\frac{\partial}{\partial x}$$

is complete. For example, we can take

$$h(x) := \begin{cases} e^{-\frac{1}{x^2}} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

The time-one flow of ξ is a diffeomorphism $\psi : \mathbb{R} \to \mathbb{R}$ such that

- its jet at 0 coincides with the jet at 0 of the identity map, and
- $\psi(\mathbb{R}_{>0}) = \mathbb{R}_{>0}$ and $\psi(\mathbb{R}_{<0}) = \mathbb{R}_{<0}$.

The inverse $\psi^{-1} : \mathbb{R} \to \mathbb{R}$ has the same properties. So does

$$\hat{\psi}(x) := \begin{cases} \psi(x) & \text{if } x \geq 0 \\ \psi^{-1}(x) & \text{if } x < 0. \end{cases}$$

Let G be a copy of $\mathbb{Z}$, with action on $\mathbb{R}$ generated by ψ. Let H be a copy of $\mathbb{Z}$, with action on $\mathbb{R}$ generated by $\hat{\psi}$. Then G and H have the same orbits, so

$$\mathbb{R}/G = \mathbb{R}/H =: \mathbb{R}/\sim.$$

Proposition 5.35. *The identity map on $\mathbb{R}/\sim$ does not lift to a biprincipal bibundle between the action groupoids $G \ltimes \mathbb{R}$ and $H \ltimes \mathbb{R}$. Moreover, these action groupoids are not Morita equivalent.*

Proof. For a contradiction, suppose these groupoids are Morita equivalent. By Example 2.24, an invertible bibundle $P : G \ltimes \mathbb{R} \to \mathbb{R} \rtimes H$ (we can view the H action as a right action because H is abelian) is given by the diagram

$$\begin{tikzcd} & G \circlearrowright P \circlearrowleft H \arrow[dl, "a"'] \arrow[dr, "a'"] & \\ G \circlearrowright \mathbb{R} & & \mathbb{R} \circlearrowleft H, \end{tikzcd}$$

where a is a principal H-bundle, and is G-equivariant, and a' is a principal G-bundle, and is H-equivariant, and the G and H actions on P commute.

Because they are bundles over $\mathbb{R}$, both a and a' are trivial. Choose a global section σ of a. By Proposition 2.32,

$$\varphi := a'\sigma : \mathbb{R} \to \mathbb{R}$$

descends to a diffeomorphism $\mathbb{R}/G \to \mathbb{R}/H$. By equivariance of a, both $\sigma(1 \cdot x)$ and $1 \cdot \sigma(x)$ are in the same a-fiber (note that 1 is the generator of the G and H action). Therefore we have the smooth map

$$\mathbb{R} \to P \times_a P, \quad x \mapsto (\sigma(1 \cdot x), 1 \cdot \sigma(x)).$$

By principality of $P \xrightarrow{a} \mathbb{R}$, this yields a smooth map $\eta : \mathbb{R} \to H$ given by

$$\sigma(1 \cdot x) = (1 \cdot \sigma(x)) \cdot \eta(x),$$

and because H is discrete, $\eta \in H$ is constant. Therefore

$$\varphi(1 \cdot x) = \varphi(x) \cdot \eta, \text{ and so } \varphi(k \cdot x) = \varphi(x) \cdot (k\eta).$$

There are two cases.

- If $\eta = 0$, then φ is G-invariant. By definition of the G action, for each x, we have $k \cdot x \to 0$ as $k \to -\infty$, therefore $\varphi(k \cdot x) = \varphi(x) \to \varphi(0)$, and thus $\varphi(x) = \varphi(0)$ for all x. But this contradicts the fact φ descends to a diffeomorphism $\mathbb{R}/G \to \mathbb{R}/H$.
- If $\eta \neq 0$, then we get a contradiction as follows: for x such that $\operatorname{sign}(\eta) = -\operatorname{sign}(\varphi(x))$,

  $$\varphi(k \cdot x) = \hat{\psi}^{k\eta}(\varphi(x)) = \psi^{\operatorname{sign}(\varphi(x))k\eta}(\varphi(x)) \to \pm\infty \text{ as } k \to -\infty$$

  On the other hand, $\varphi(k \cdot x) \to \varphi(0) = 0$ as $k \to -\infty$, so we have our contradiction.

$\square$

Furthermore,

Proposition 5.36. $\mathbb{R}/\sim$ *is not a diffeological quasifold.*

Proof. Seeking a contradiction, suppose $\mathbb{R}/\sim$ is a diffeological quasifold. Take an open neighbourhood U of $[0]$ in X, a countable subgroup Γ of $\mathrm{Aff}(\mathbb{R})$, a Γ-invariant open subset V of $\mathbb{R}$, and a diffeomorphism $F : U \to \mathsf{V}/\Gamma$. Fix some $x_0 \in \mathsf{V}$ such that $F([0]) = [x_0]$.

By the exact same argument from the first half of the proof of Lemma 5.9, we may find locally defined lifts $f, s : \mathbb{R} \to \mathbb{R}$ of F and F^{-1}, respectively, such that $f(0) = x_0$ and $s(x_0) = 0$. Then fs is a locally defined map that preserves Γ orbits, and fixes x_0.

The composition $f\psi s$ is then also a locally defined map that preserves Γ orbits and fixes x_0. Fix an open interval W about x_0 such that both fs and $f\psi s$ are defined on W.

By Lemma 5.7, there are $\gamma, \delta \in \Gamma$ such that $fs = \gamma$ and $f\psi s = \delta$ on W. Therefore $(fs)' = \gamma'$, and in particular s' is non-vanishing on W. It follows that s is a diffeomorphism on a neighbourhood of x_0. Similarly, f' does not vanish on $s(\mathrm{W})$, and so f is a diffeomorphism on a neighbourhood of 0.

Since $\psi'(0) = 1$,

$$\gamma' = (fs)'(x_0) = (f\psi s)'(x_0) = \delta'.$$

Thus γ and δ differ only by a translation; because $\psi(0) = 0$, we see that $\gamma = \delta$. In other words, $fs = f\psi s$ on W. Since f and s are diffeomorphisms, we conclude that $\psi = \mathrm{id}$ in a neighbourhood of 0, which contradictions the definition of ψ. Therefore no such diffeomorphism F exists, and $\mathbb{R}/\sim$ is not a diffeological 1-quasifold. □

We suspect that the key feature of the G action on $\mathbb{R}$ allowing for Propositions 5.35 and 5.36 is that the G-action is not *jet-determined*; all the ψ^k have the same jet at 0, yet are different diffeomorphisms for different k. This is what allows for the existence of the smooth function $\hat{\psi}$. Conversely, given a jet-determined action of a discrete group Γ on $\mathbb{R}^n$, at the time of writing we do not know if a lemma analogous to Lemma 5.7 holds.

The G action on $\mathbb{R}$ also gives rise to a counter-example in foliation theory. We will assume familiarity with foliations and their holonomy.

Proposition 5.37. *There exists a foliated manifold $(M, \mathcal{F})$ for which the quotient space $M/\mathcal{F}$ is not a diffeological quasifold.*

Proof. Consider $\mathbb{R}^2$ with coordinates (t, x), equipped with the foliation $\mathcal{F}$ spanned by $\frac{\partial}{\partial t}$. The leaves of $\mathcal{F}$ are the horizontal lines $\mathbb{R} \times \{x\}$. We have a $\mathbb{Z}$ action on $\mathbb{R}^2$ given by

$$k \cdot (t, x) := (t + k, \psi^k(x)),$$

where ψ is the flow from the beginning of this section. This action is free, properly discontinuous, and preserves $\mathcal{F}$. Therefore $M := \mathbb{R}^2/\mathbb{Z}$ is a manifold, the quotient $\pi : \mathbb{R}^2 \to M$ is a covering map, and $\mathcal{F}/\mathbb{Z} := d\pi(\mathcal{F})$ is a foliation on M. The leaves of $\mathcal{F}/\mathbb{Z}$ are of the form $\pi\left(\mathbb{R} \times \{x\}\right)$, for $x \in \mathbb{R}$. A connected total transversal to $\mathcal{F}/\mathbb{Z}$ is given by $T := \pi(\{0\} \times \mathbb{R}) \cong \mathbb{R}$.

The holonomy pseudogroup on T associated to $\mathcal{F}/\mathbb{Z}$ is the pseudogroup generated by the group $\{\psi^k\}$. In particular, the quotient $M/(\mathcal{F}/\mathbb{Z})$ is diffeologically diffeomorphic to $T/\{\psi^k\}$.[5] But as we have seen in Proposition 5.35, $T/\{\psi^k\}$ is not a diffeological quasifold. Therefore neither is $M/(\mathcal{F}/\mathbb{Z})$. □

5.5 FUTURE DIRECTIONS

Proposition 5.37 raises the following question: for which foliations $\mathcal{F}$ is the leaf space $M/\mathcal{F}$ a diffeological quasifold? Some candidates include the null foliation on a compact, connected presymplectic toric manifold, in the sense of Ratiu and Zung [RZ19] or Lin and Sjamaar [LS19], or *Riemannian* foliations, as defined in [Mol88]. This question is the subject of work in progress together with Yi Lin [LM23].

Whether or not leaf spaces of Riemannian foliations are diffeological quasifolds, we expect Theorem 5.32 to generalize to these spaces. This is because such leaf spaces have the form T/Ψ, where T is a manifold (for instance, a complete transversal), and Ψ is a countably-generated pseudogroup (for instance, the holonomy pseudogroup of $\mathcal{F}$) satisfying:

Property. *(Lift) Whenever U is an open subset of T, and $f : U \to T$ is a smooth map that preserves Ψ orbits, f is in Ψ.*

Compare this to Lemma 5.7. In work in preparation [Miy23b], we show that Theorem 5.32 extends to the categories of étale effective Lie groupoids $\Gamma \rightrightarrows T$ whose pseudogroups Ψ satisfy Property (Lift), and to diffeological spaces of the form T/Ψ. These include diffeological quasifolds and quasifold groupoids, and also the étale Holonomy groupoids of complete Riemannian foliations. Moreover in [Miy23b], we also show that we can expand the collection of arrows to include submersions between Lie groupoids, in the sense of [dHLF19], and local subductions between diffeological spaces (cf. Definition 2.8, and see [IZ13, Article 2.16]).

5 This is because $T/\{\psi^k\}$ is the orbit space of the pullback of the holonomy groupoid associated to $\mathcal{F}/\mathbb{Z}$ to the transversal T, and these groupoids are Morita equivalent.

BIBLIOGRAPHY

[ABT13] Marcos M. Alexandrino, Rafael Briquet, and Dirk Töben, *Progress in the theory of singular Riemannian foliations*, Differential Geom. Appl. **31** (2013), no. 2, 248–267. MR 3032647

[AS09] Iakovos Androulidakis and Georges Skandalis, *The holonomy groupoid of a singular foliation*, J. Reine Angew. Math. **626** (2009), 1–37. MR 2492988

[AZ13] Iakovos Androulidakis and Marco Zambon, *Smoothness of holonomy covers for singular foliations and essential isotropy*, Math. Z. **275** (2013), no. 3-4, 921–951. MR 3127043

[AZ16] ———, *Stefan-Sussmann singular foliations, singular subalgebroids and their associated sheaves*, Int. J. Geom. Methods Mod. Phys. **13** (2016), no. suppl., 1641001, 17. MR 3556105

[BP18] Fiammetta Battaglia and Elisa Prato, *Nonrational symplectic toric cuts*, Internat. J. Math. **29** (2018), no. 10, 1850063, 19. MR 3861905

[BP19] ———, *Nonrational symplectic toric reduction*, J. Geom. Phys. **135** (2019), 98–105. MR 3872625

[BPZ19] Fiammetta Battaglia, Elisa Prato, and Dan Zaffran, *Hirzebruch surfaces in a one-parameter family*, Boll. Unione Mat. Ital. **12** (2019), no. 1-2, 293–305. MR 3936308

[BZ15] Fiammetta Battaglia and Dan Zaffran, *Foliations modeling nonrational simplicial toric varieties*, Int. Math. Res. Not. IMRN (2015), no. 22, 11785–11815. MR 3456702

[CdHP20] Alejandro Cabrera, Matias del Hoyo, and Enrique Pujals, *Discrete dynamics and differentiable stacks*, Rev. Mat. Iberoam. **36** (2020), no. 7, 2121–2146. MR 4163995

[Cle66] Alfred Clebsch, *Ueber die simultane Integration linearer partieller Differentialgleichungen*, J. Reine Angew. Math. **65** (1866), 257–268. MR 1579322

[CS13] Marius Crainic and Ivan Struchiner, *On the linearization theorem for proper Lie groupoids*, Ann. Sci. Éc. Norm. Supér. (4) **46** (2013), no. 5, 723–746. MR 3185351

[Dea40] Feodor Deahna, *Ueber die Bedingungen der Integrabilität lineärer Differentialgleichungen erster Ordnung zwischen einer beliebigen Anzahl veränderlicher Größen*, J. Reine Angew. Math. **20** (1840), 340–349. MR 1578247

[dHLF19] Matias del Hoyo and Rui Loja Fernandes, *Riemannian metrics on differentiable stacks*, Math. Z. **292** (2019), no. 1-2, 103–132. MR 3968895

[DI85] Paul Donato and Patrick Iglésias, *Exemples de groupes difféologiques: flots irrationnels sur le tore*, C. R. Acad. Sci. Paris Sér. I Math. **301** (1985), no. 4, 127–130. MR 799609

[DLPR12] Lance D. Drager, Jeffrey M. Lee, Efton Park, and Ken Richardson, *Smooth distributions are finitely generated*, Ann. Global Anal. Geom. **41** (2012), no. 3, 357–369. MR 2886203

[Fer15] Rui Loja Fernandes, *Normal forms and Lie groupoid theory*, Geometric methods in physics, Trends Math., Birkhäuser/Springer, Cham, 2015, pp. 49–66. MR 3629663

[Fro77] G. Frobenius, *Ueber das Pfaffsche Problem*, J. Reine Angew. Math. **82** (1877), 230–315. MR 1579710

[GZ19] Alfonso Garmendia and Marco Zambon, *Hausdorff Morita equivalence of singular foliations*, Ann. Global Anal. Geom. **55** (2019), no. 1, 99–132. MR 3916125

[Haw05] Thomas Hawkins, *Frobenius, Cartan, and the problem of Pfaff*, Arch. Hist. Exact Sci. **59** (2005), no. 4, 381–436. MR 2188939

[Her62] Robert Hermann, *The differential geometry of foliations. II*, J. Math. Mech. **11** (1962), 303–315. MR 0142131

[HM04] Andre Henriques and David S. Metzler, *Presentations of noneffective orbifolds*, Trans. Amer. Math. Soc. **356** (2004), no. 6, 2481–2499. MR 2048526

[HMVSC11] Gilbert Hector, Enrique. Macías-Virgós, and Esperanza. Sanmartín-Carbón, *De Rham cohomology of diffeological spaces and foliations*, Indag. Math. (N.S.) **21** (2011), no. 3-4, 212–220. MR 2835929

[Hof20] Benjamin Hoffman, *Toric symplectic stacks*, Adv. Math. **368** (2020), 107135, 43. MR 4082991

[HS21] Benjamin Hoffman and Reyer Sjamaar, *Stacky Hamiltonian actions and symplectic reduction*, Int. Math. Res. Not. IMRN (2021), no. 20, 15209–15300. MR 4329869

[IKZ10] Patrick Iglesias, Yael Karshon, and Moshe Zadka, *Orbifolds as diffeologies*, Trans. Amer. Math. Soc. **362** (2010), no. 6, 2811–2831. MR 2592936

[IZ13] Patrick Iglesias-Zemmour, *Diffeology*, Mathematical Surveys and Monographs, vol. 185, American Mathematical Society, Providence, RI, 2013. MR 3025051

[IZ22] ———, *Diffeology*, Beijing World Publication Corporation, 2022, Reprint of IZ13.

[IZL18] Patrick Iglesias-Zemmour and Jean-Pierre Laffineur, *Noncommutative geometry and diffeology: the case of orbifolds*, J. Noncommut. Geom. **12** (2018), no. 4, 1551–1572. MR 3896235

[IZP21] Patrick Iglesias-Zemmour and Elisa Prato, *Quasifolds, diffeology and noncommutative geometry*, J. Noncommut. Geom. **15** (2021), no. 2, 735–759. MR 4325720

[Jac27] Carl G. J. Jacobi, *Ueber die Integration der partiellen Differentialgleichungen erster Ordnung*, J. Reine Angew. Math. **2** (1827), 317–329. MR 1577659

[Jor82] Henri Joris, *Une $\mathcal{C}^\infty$-application non-immersive qui possède la propriété universelle des immersions*, Arch. Math. (Basel) **39** (1982), no. 3, 269–277. MR 682456

[KM22] Yael Karshon and David Miyamoto, *Quasifold groupoids and diffeological quasifolds*, Submitted preprint. https://arxiv.org/abs/2206.14776, 2022.

[KMW22] Yael Karshon, David Miyamoto, and Jordan Watts, *Diffeological submanifolds and their friends*, Submitted preprint. https://arxiv.org/abs/2204.10381, 2022.

[Kos53] Jean-Louis Koszul, *Sur certains groupes de transformations de Lie.*, Géométrie différentielle. Colloques Internationaux du Centre National de la Recherche Scientifique, Strasbourg, 1953, Centre National de la Recherche Scientifique, Paris, 1953, pp. 137–141. MR 59919

[Kub90] Jan Kubarski, *About Stefan's definition of a foliation with singularities: a reduction of the axioms*, Bull. Soc. Math. France **118** (1990), no. 4, 391–394. MR 1090406

[KW16] Yael Karshon and Jordan Watts, *Basic forms and orbit spaces: a diffeological approach*, SIGMA Symmetry Integrability Geom. Methods Appl. **12** (2016), Paper No. 026, 19. MR 3470746

[Lav18] Sylvain Lavau, *A short guide through integration theorems of generalized distributions*, Differential Geom. Appl. **61** (2018), 42–58. MR 3856749

[Lee13] John M. Lee, *Introduction to smooth manifolds*, second ed., Graduate Texts in Mathematics, vol. 218, Springer, New York, 2013. MR 2954043

[Ler10] Eugene Lerman, *Orbifolds as stacks?*, Enseign. Math. (2) **56** (2010), no. 3-4, 315–363. MR 2778793

[LM23] Yi Lin and David Miyamoto, *Riemannian foliations and quasifolds*, Manuscript in progress, 2023.

[Lob70] Claude Lobry, *Contrôlabilité des systèmes non linéaires.*, SIAM J. Control (1970), 573–605. MR 271979

[LS19] Yi Lin and Reyer Sjamaar, *Convexity properties of presymplectic moment maps*, J. Symplectic Geom. **17** (2019), no. 4, 1159–1200. MR 4031537

[Mat68] Michihiko Matsuda, *An integration theorem for completely integrable systems with singularities*, Osaka Math. J. **5** (1968), 279–283. MR 243555

[McD19] Dusa McDuff, *Notes on Kuranishi atlases*, Virtual fundamental cycles in symplectic topology, Math. Surveys Monogr., vol. 237, Amer. Math. Soc., Providence, RI, 2019, pp. 1–109. MR 3931094

[Mei03] Eckhard Meinrenken, *Group actions on manifolds*, Lecture notes, 2003.

[Miy23a] David Miyamoto, *The Basic de Rham Complex of a Singular Foliation*, Int. Math. Res. Not. IMRN (2023), no. 8, 6364–6401. MR 4574377

[Miy23b] ______, *Foliations determined by their orbit spaces (tentative)*, 2023, Manuscript in preparation.

[Miy23c] ———, *Singular foliations through diffeology*, Accepted preprint. https://arxiv.org/abs/2303.07494. To appear in Contemporary Mathematics (proceedings) for the special session "Recent advances on diffeologies and their applications" held at the AMS-SMF-EMS Joint International Meeting in Genoble, July 2022, published by the American Mathematical Society, 2023.

[MM03] Ieke Moerdijk and Janez Mrčun, *Introduction to foliations and Lie groupoids*, Cambridge Studies in Advanced Mathematics, vol. 91, Cambridge University Press, Cambridge, 2003. MR 2012261

[Mol88] Pierre Molino, *Riemannian foliations*, Progress in Mathematics, vol. 73, Birkhäuser Boston, Inc., Boston, MA, 1988, Translated from the French by Grant Cairns, With appendices by Cairns, Y. Carrière, É. Ghys, E. Salem and V. Sergiescu. MR 932463

[MR19] Ricardo A. E. Mendes and Marco Radeschi, *A slice theorem for singular Riemannian foliations, with applications*, Trans. Amer. Math. Soc. **371** (2019), no. 7, 4931–4949. MR 3934473

[Nag66] Tadashi Nagano, *Linear differential systems with singularities and an application to transitive Lie algebras*, J. Math. Soc. Japan **18** (1966), 398–404. MR 199865

[Pal61] Richard S. Palais, *On the existence of slices for actions of non-compact Lie groups*, Ann. of Math. (2) **73** (1961), 295–323. MR 126506

[Pra99] Elisa Prato, *Sur une généralisation de la notion de V-variété*, C. R. Acad. Sci. Paris Sér. I Math. **328** (1999), no. 10, 887–890. MR 1689861

[Pra01] ———, *Simple non-rational convex polytopes via symplectic geometry*, Topology **40** (2001), no. 5, 961–975. MR 1860537

[PTW21] Hessel Posthuma, Xiang Tang, and Kirsten Wang, *Resolutions of proper Riemannian Lie groupoids*, Int. Math. Res. Not. IMRN (2021), no. 2, 1249–1287. MR 4201416

[Rei59] Bruce L. Reinhart, *Foliated manifolds with bundle-like metrics*, Ann. of Math. (2) **69** (1959), 119–132. MR 107279

[RZ19] Tudor Ratiu and Nguyen Tien Zung, *Presymplectic convexity and (ir)rational polytopes*, J. Symplectic Geom. **17** (2019), no. 5, 1479–1511. MR 4039815

[Sam01] Hans Samelson, *Differential forms, the early days; or the stories of Deahna's theorem and of Volterra's theorem*, Amer. Math. Monthly **108** (2001), no. 6, 522–530. MR 1840658

[Ste74] Peter Stefan, *Accessible sets, orbits, and foliations with singularities*, Proc. London Math. Soc. (3) **29** (1974), 699–713. MR 362395

[Sus73] Héctor J. Sussmann, *Orbits of families of vector fields and integrability of distributions*, Trans. Amer. Math. Soc. **180** (1973), 171–188. MR 321133

[Sus08] ———, *Smooth distributions are globally finitely spanned*, Analysis and design of nonlinear control systems, Springer, Berlin, 2008, pp. 3–8. MR 2758930

[Wan17] Roy Wang, *On integrable systems & rigidity for pdes with symmetry*, Ph.D. thesis, Utrecht University, Utrecht, NL, 2017, PhD Thesis.

[Wat17] Jordan Watts, *The differential structure of an orbifold*, Rocky Mountain J. Math. **47** (2017), no. 1, 289–327. MR 3619765

[Wat22] ———, *The orbit space and basic forms of a proper Lie groupoid*, Current trends in analysis, its applications and computation, Trends Math. Res. Perspect., Birkhäuser/Springer, Cham, 2022, pp. 513–523. MR 4559547

[Wei02] Alan Weinstein, *Linearization of regular proper groupoids*, J. Inst. Math. Jussieu **1** (2002), no. 3, 493–511. MR 1956059

[Zog10] Masrour Zoghi, *The Gromov Width of Coadjoint Orbits of Compact Lie Groups*, ProQuest LLC, Ann Arbor, MI, 2010, Thesis (Ph.D.)–University of Toronto (Canada). MR 2873388

[Zun06] Nguyen Tien Zung, *Proper groupoids and momentum maps: linearization, affinity, and convexity*, Ann. Sci. École Norm. Sup. (4) **39** (2006), no. 5, 841–869. MR 2292634

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