RESEARCH STATEMENT

The category $\mathcal{M}$fld of smooth finite-dimensional manifolds is the basic setting for differential geometry. It is notoriously rigid, with nice objects but poor properties: subsets and quotients of, and mapping spaces between, manifolds are rarely themselves manifolds. These deficiencies obstruct several important results, such as

- Symplectic reduction: given a Hamiltonian action of a Lie group G on a symplectic manifold (M, ω), with a momentum map $\mu\colon M \to \mathfrak{g}^*$, the reduced space $M_{\mathrm{red}} := \mu^{-1}(0)/G$ is a symplectic manifold if and only if the G-action is free and proper.
- Infinite-dimensional Lie theory: if M is a non-compact manifold, there is no (infinite-dimensional) manifold structure for which the Lie algebra of $\mathrm{Diff}(M)$ is isomorphic to the Lie algebra of vector fields $\mathfrak{X}(M)$.

Diffeological spaces, introduced by Souriau [Sou80], generalize manifolds. In sophisticated terms, a diffeological space is a concrete sheaf on the site $\mathcal{O}$pen of open subsets of Cartesian spaces. In grounded terms, a diffeology on a set X is a collection of maps $\mathscr{D} = \{U \xrightarrow{p} X \mid U \in \mathcal{O}\mathrm{pen}\}$ which contains all constant maps and satisfies the sheaf axioms. The category $\mathcal{D}$flg of diffeological spaces is geometric, and is closed under taking subsets, quotients, and mapping spaces.(1) We could now ask, for instance, "is M_{red} always a symplectic diffeological space, with symplectic form induced by ω?" or "is $\mathrm{Lie}(\mathrm{Diff}(M)) \cong \mathfrak{X}(M)$, and what is Lie?"

At the same time, there are other generalizations of smooth manifolds that can address such questions. Taking a cue from algebraic geometry and moving to higher categorical settings, one can use orbifolds, and more generally differentiable stacks, to model quotients of manifolds. Taking a cue from functional analysis, one may consider infinite-dimensional Banach, Fréchet, or convenient(2) manifolds to handle mapping spaces.

My research combines the infinite-dimensional, higher geometric, and diffeological approaches to generalize those qualified theorems in $\mathcal{M}$fld, or its infinite-dimensional incarnations, beyond their obstructions. The results and forecasts of these efforts are captured by the specific examples in the sequel. Relevant MSC classifications are 58A40, 22E65, 18F40, 57R30, and 58H05.

1. Infinite-dimensional Lie theory

Let us begin with the well-known functor

$$\mathrm{Lie}\colon \{\text{Lie groups}\} \to \{\text{Lie algebras}\}, \quad G \mapsto T_e G.$$

Defining Lie traditionally requires the tangent functor $\hat{T}\colon \mathcal{M}\mathrm{fld} \to \mathcal{M}\mathrm{fld}$, and a Lie algebra structure on $\mathfrak{X}(M)$ that is typically given by identifying it with the algebra of derivations on $C^\infty(M, \mathbb{R})$. But this identification does not generalize to diffeological spaces; many spaces that we require(3) to have non-trivial Lie algebras, such as irrational tori $\mathbb{R}/(\mathbb{Z} + \alpha\mathbb{Z})$ for irrational α, carry only trivial derivations. Thus alternatively, we realize that we can recover the Lie bracket on $\mathfrak{X}(M)$ using the entire *tangent structure* on M, in the sense of Rosický [Ros84]: this is the data of T, together with

(1)It is a quasitopos.

(2)In the sense of Kriegl and Michor.

(3)We will see why shortly.

the natural transformations

$$\begin{array}{ll} \text{the projection} & \pi\colon T \to 1 \\ \text{the zero-section} & 0\colon 1 \to T \\ \text{the addition} & +\colon T_2 \to T, \text{ where } T_2M := TM \times_\pi TM \\ \text{the vertical lift} & \lambda\colon T \to T^2 \\ \text{the canonical flip} & \tau\colon T^2 \to T^2, \end{array}$$

satisfying a list of axioms. For $\mathcal{D}$flg, we begin by choosing the tangent functor

$$T := \mathrm{Lan}_y(y \circ [\hat{T}\colon \mathcal{M}\mathrm{fld} \to \mathcal{M}\mathrm{fld}]),$$

the left Kan extension of $\hat{T}\colon \mathcal{M}\mathrm{fld} \to \mathcal{M}\mathrm{fld}$ along the embedding $y\colon \mathcal{M}\mathrm{fld} \to \mathcal{D}\mathrm{flg}$. In pioneering work towards diffeological Lie theory, Blohmann isolated a full subcategory $\mathcal{E}$last of $\mathcal{D}$flg, whose objects are called *elastic* spaces, for which:

Theorem (Blohmann [Blo24])**.** *The category $\mathcal{E}$last, equipped with T and (essentially)*[(4)] *the left Kan extensions of all the tangent structure data on $\mathcal{M}$fld, is a tangent category.*

As a consequence, elastic diffeological groups G have elastic Lie algebras $\mathfrak{g} = T_eG$, with Lie algebra structure induced by the Lie bracket on left-invariant vector fields on G.

Question 1. For the functor

$$\mathrm{Lie}\colon \{\text{elastic groups}\} \to \{\text{elastic Lie algebras}\},$$

- Does every elastic Lie algebra integrate to an elastic group?
- Are "simply-connected" integrations universal?
- How do Lie algebra morphisms lift to elastic group morphisms?

This instantiates a program. In joint work in-progress with Christian Blohmann, we are pursuing the first point. We have proved:

Theorem 2. *If M is compact, the group* $\mathrm{Diff}(M)$ *is elastic, with Lie algebra isomorphic to* $\mathfrak{X}(M)$*.*

Our goal is to show the same is true for M non-compact. This lies outside the realm of infinite-dimensional manifolds, where $\mathrm{Diff}(M)$ is modelled on the space of vector fields of compact support. We have thus far showed that $\mathfrak{X}(\mathbb{R}^n)$ is a quotient of $T_{\mathrm{id}}\,\mathrm{Diff}(\mathbb{R}^n)$.

Even within the realm of infinite-dimensional geometry, elastic groups clarify and extend Lie theory. In finite-dimensions, we have Lie's celebrated third theorem.

Theorem (Lie)**.** *For every finite-dimensional Lie algebra $\mathfrak{g}$, there is a Lie group G with* $\mathrm{Lie}(G) = \mathfrak{g}$.

Cartan constructed G by integrating central extensions of Lie algebras. Let $\mathfrak{z}$ denote the center of $\mathfrak{g}$, and let $\mathfrak{g}_{\mathrm{ad}}$ denote $\mathfrak{g}/\mathfrak{z}$. The latter integrates to a simply-connected Lie group G_{ad}, since it acts faithfully on $\mathfrak{g}$. The former integrates to $(\mathfrak{z}, +)$, or its quotient by any topologically discrete subgroup Π. We seek G fitting into the diagram:

$$\begin{tikzcd} \{e\} \arrow[r] \arrow[d, dashed, "\mathrm{Lie}"] & \mathfrak{z}/\Pi \arrow[r] \arrow[d, dashed, "\mathrm{Lie}"] & G \arrow[r] \arrow[d, dashed, "\mathrm{Lie}"] & G_{\mathrm{ad}} \arrow[r] \arrow[d, dashed, "\mathrm{Lie}"] & \{e\} \arrow[d, dashed, "\mathrm{Lie}"] \\ 0 \arrow[r] & \mathfrak{z} \arrow[r] & \mathfrak{g} \arrow[r] & \mathfrak{g}_{\mathrm{ad}} \arrow[r] & 0. \end{tikzcd} \tag{1}$$

The extension of Lie algebras is completely determined by the Lie algebra 2-cocycle $\omega\colon \mathfrak{g}_{\mathrm{ad}} \times \mathfrak{g}_{\mathrm{ad}} \to \mathfrak{z}$, which extracts the $\mathfrak{z}$-component of the Lie bracket. This, in turn, determines the *group of periods*

[(4)] The actual tangent structure data are not all Kan extensions.

$\Pi_\omega \leq \mathfrak{z}$, which arises as the image of $\pi_2(G_{\mathrm{ad}})$ under a group homomorphism. If Π_ω is topologically discrete, then we may construct a G that completes the diagram (1), with $\Pi = \Pi_\omega$. When $\mathfrak{g}$ is finite-dimensional, the group G_{ad} is a Lie group, and so $\pi_2(G_{\mathrm{ad}})$, hence also Π_ω, is trivial. Thus we have integrated $\mathfrak{g}$ without leaving $\mathcal{M}$fld.

This fails in infinite dimensions: there are examples Banach Lie groups for which Π_ω is not topologically discrete (the first was given in [EK64]). However, this is not an obstruction for diffeology. I have proved[(5)]

Theorem 3 ([Miy25])**.** *Let $\mathfrak{g}$ be a Banach Lie algebra, whose center $\mathfrak{z}$ is complemented. The integration of the central extension $\mathfrak{z} \to \mathfrak{g} \to \mathfrak{g}_{ad}$ as indicated by (1) exists in $\mathcal{E}$lst if $\pi_2(G_{ad})$ is countable.* A fortiori*, $\mathfrak{g}$ integrates to a diffeological group.*

Remark 4. Implicit in the statement of this theorem is the fact, also proved in [Miy25], that the category of Banach (more generally Fréchet or convenient) manifolds embeds into $\mathcal{E}$lst.

The countability of $\pi_2(G_{\mathrm{ad}})$ ensures that Π_ω is diffeologically discrete; the quotients $\mathfrak{z}/\Pi_\omega$ are infinite-dimensional irrational tori. Integration to stacks, without an assumption on $\pi_2(G_{\mathrm{ad}})$, was carried out in [Woc11; WZ16]. Formalizing the connection between the stacky and diffeological integrations of $\mathfrak{g}$, and resolving the diffeological integration of $\mathfrak{g}$ without assumption on $\pi_2(G_{\mathrm{ad}})$, constitute two avenues for future research.

1.1. **Integrating Lie algebroids and singular foliations.** Lie groupoids and Lie algebroids are "horizontal" generalizations of Lie groups and Lie algebras. Examples include Lie group and Lie algebra bundles. We again have a Lie functor:

$$\mathrm{Lie} \coloneqq \{\text{Lie groupoids}\} \to \{\text{Lie algebroids}\}, \quad \mathcal{G} \mapsto A.$$

Non-integrable Lie algebroids were found by [AM85]. Crainic and Fernandes [CF03] gave necessary and sufficient conditions for integrability. When A is transitive over a simply-connected base, there is an associated "period group" $\Pi(A)$ identifying with a proper subgroup of $(\mathbb{R}, +)$, and

Theorem (Crainic-Fernandes)**.** *A is integrable if and only if $\mathbb{R}/\Pi(A)$ is a manifold.*

Once again, $\mathbb{R}/\Pi(A)$ is an elastic diffeological space, so we pose the question:

Question 5. Does every transitive Lie algebroid integrate to an elastic diffeological groupoid?

But we have begged the question:

Question 6. Is there a Lie functor

$$\mathrm{Lie} : \{\text{elastic Lie groupoids}\} \to \{\text{elastic Lie algebroids}\}?$$

What constitutes the right-hand side?

This instantiates another program. Blohmann and Aintablian's work [AB24] proposes a Lie functor. With other tools, Villatoro [Vil23] introduced a Lie functor for yet another class of diffeological groupoids, and with this integrated general Lie algebroids. My future research will include establishing integration of transitive Lie algebroids into elastic groupoids, and joining the elastic integration with Villatoro's.

Underlying Lie groupoids and Lie algebroids are singular foliations. A leaf-first approach (cf. [Miy24]) defines a singular foliation $\mathcal{F}$ as a partition of a manifold M into connected weakly-embedded submanifolds (leaves) of varying dimension, that fit together in a smooth way. Another approach (cf. [AS09]) defines a singular foliation as an involutive and locally finitely generated submodule $\mathcal{F}$ of compactly supported vector fields $\mathfrak{X}_c(M)$. The latter induces the former.

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(5) Answering a conjecture of Wockel [Woc11, Remark 7.1]

When $\mathcal{F}$ is a regular foliation, it integrates to the holonomy groupoid $\mathrm{Hol}(clF) \rightrightarrows M$. This is a Lie groupoid, and with it we may view the leaf space $M/\mathcal{F}$ as a differentiable stack. For a general singular foliation, Androulidakis and Skandalis [AS09] defined a topological groupoid $\mathrm{Hol}(\mathcal{F}) \rightrightarrows M$, which is rarely Lie. However, it carries a natural diffeology.

Question 7. In what sense does the diffeological groupoid $\mathrm{Hol}(\mathcal{F})$ differentiate to $\mathcal{F}$?

In joint work in progress with Leonid Ryvkin and Alfonso Garmendia, we define a class of diffeological groupoids that includes $\mathrm{Hol}(\mathcal{F})$, and that we expect supports an appropriate Lie functor. These are defined via *tepui* fibrations $X \to M$, of which the simplest arise as quotients of vector bundles (see [GMR25, Lemma 1.37]):

Example 8. Let $V \to M$ be a vector bundle, and $D \subseteq V$ be a smooth singular subbundle.[6] Then $V/D \to M$ is a tepui fibration and a diffeological vector bundle, called a VB-tepui.

Consonant with this proposal's theme, we can extend the classical Serre-Swan theorem:

Theorem 9 ([GMR25])**.** *For a fixed manifold M, the global section functor*

$$\Gamma\colon \{\textit{VB-tepui over } M\} \to \{\textit{finite-type, global, and fiber-determined } C^\infty(M)\textit{-modules}\}$$

is an equivalence of categories.

This result assigns to $\mathcal{F}$ a canonical tepui algebroid. Differentiating $\mathrm{Hol}(\mathcal{F})$ to this algebroid is ongoing work.

2. Quasi-étale spaces and Q-manifolds

Several of the quotients above have the same property. What follows is contained in [Miy24].

Definition 10. A smooth action of a Lie group G on a manifold M is called *lift-complete* if the only local lifts of the identity on M/G are given by the G-action. It furthermore has property (Q) if every plot $U \to M/G$ has a unique lift up to germ.

I generalize lift-completeness, and property (Q), to étale effective Lie groupoids $G \rightrightarrows M$.

Definition 11. A diffeological space is *quasi-étale* if it is isomorphic to the orbit space of some lift-complete Lie groupoid. It furthermore is a *Q-manifold* if the corresponding Lie groupoid has property (Q).

Example 12.

- With Yael Karshon [KM23], I proved that quasifolds (cf. [Pra01; IP21]) are quasi-étale. Quasifolds appear in Prato's generalization of the Delzant theorem in symplectic geometry.
- I proved in [Miy24] that the leaf space of any Riemannian foliation is quasi-étale.
- With Yi Lin [LM24], we showed that most *Killing* Riemannian foliations are quasifolds. In particular, this applies to quotients G/H of Lie groups.

Lift-complete Lie groupoids and quasi-étale spaces give a setting where the higher geometric and diffeological approaches coincide.

Theorem 13. *The natural quotient functor taking a Lie groupoid to its diffeological orbit space restricts to an equivalence of: the category of étale effective lift-complete Lie groupoids, with morphisms (isomorphism classes of) surjective submersive bibundles, and the category of quasi-étale diffeological spaces, with morphisms surjective local subductions. If we impose property (Q), then we do not need to restrict the morphisms.*

This directly generalizes [KM23]. For property (Q), we have an equivalence of categories.

[6] A subbundle with fibers of varying dimension

Question 14. Does Theorem 13 generalize to infinite-dimensional manifolds?

A positive answer would justify why the integration in Theorem 3 does not require higher structures. It would also provide a bridge to Beltita and Pelletier's work [BP24a; BP24b] on integrating Lie algebras and Lie algebroids, which involves infinite-dimensional variants of Q-manifolds.

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