Characterizing $U(1,1)$ and Translation-Invariant Generalized Convex Valuations on $\mathbb{C}^2$

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1 A Summary of Valuation Theory

1.1 Convex valuations

Let V be an n-dimensional real vector space, and $\mathcal{K}(V)$ be the set of convex, compact, non-empty subsets of V, hereafter refered to as convex bodies. A **convex valuation** is a complex-valued function ϕ on $\mathcal{K}(V)$ such that

$$\phi(A \cup B) = \phi(A) + \phi(B) - \phi(A \cap B)$$

when $A \cup B$ and $A \cap B$ are in $\mathcal{K}(V)$. We may omit the qualifier "convex" if it is implied by context. Consider an auxillary Euclidean metric on V. Then we can endow $\mathcal{K}(V)$ with the Hausdorff metric, making $\mathcal{K}(V)$ a locally compact metric space. Changing the Euclidean structure on V produces an equivalent Hausdorff metric on $\mathcal{K}(V)$, and therefore we may unambiguously say a convex valuation ϕ is **continuous** if it is continuous in a Hausdorff metric.

A convex valuation ϕ is **translation-invariant** if $\phi(x + A) = \phi(A)$ for all $x \in V$ and $A \in \mathcal{K}(V)$. We let $\mathbf{Val}\,(\boldsymbol{V})$ denote the collection of continuous and translation-invariant convex valuations. It is a subspace of $C(\mathcal{K}(V))$, and we equip both spaces with the compact-open topology, which - since the maps are complex-valued - is also the topology of uniform convergence on compact sets. This makes $\mathrm{Val}(V)$ is a Fréchet space.[1] In fact $\mathrm{Val}(V)$ is a

[1]Note it is not necessarily saparable.

Banach space, with norm $\|\phi\| := \sup_{A \subseteq D} |\phi(A)|$, where D is the unit ball of some auxillary Euclidean metric. That the supremum is finite follows from the Blaschke selection principle, which implies the supremum is taken over a compact set in $\mathcal{K}(V)$. The triangle inequality is a consequence of McMullen's decomposition, which is stated below.

We say a convex valuation ϕ is ***k*-homogeneous** if $\phi(tA) = t^k \phi(A)$ for all $t \geq 0$. We say ϕ is **even** if $\phi(A) = \phi(-A)$, and **odd** if $\phi(-A) = -\phi(A)$. The subspaces of k-homogeneous and even/odd valuations are denoted by $\mathbf{Val}_{\boldsymbol{k}}\boldsymbol{(V)}$ and $\mathbf{Val}^{\pm}\boldsymbol{(V)}$, respectively. McMullen proved we have the direct sum decomposition

$$\mathrm{Val}(V) = \bigoplus_{k=0}^{n} \mathrm{Val}_k(V) = \bigoplus_{k=0}^{n} \mathrm{Val}_k^+(V) \oplus \mathrm{Val}_k^-(V),$$

as a corollary to his 1977 theorem:

Theorem 1 (McMullen, [16]). *Given* $\phi \in \mathrm{Val}(V)$, *and* $A_1, \ldots, A_r \in \mathcal{K}(V)$, *the complex-valued function* f *on* $(\mathbb{R}_{\geq 0})^r$ *defined by*

$$f(t_1, \ldots, t_r) := \phi(t_1 A_1 + \cdots + t_r A_r)$$

is a polynomial of degree at most $\dim(V)$.

In the case $V = \mathbb{R}^n$ with ϕ taken to be the usual volume, the coefficients of the polynomial from McMullen's theorem are Minkowski's mixed volumes. Note $\mathrm{Val}_0(V)$ and $\mathrm{Val}_n(V)$ are one-dimensional. The former is spanned by the Euler characteristic χ, which is identically 1 on $\mathcal{K}(V)$. The latter was described by Hadwiger in 1957:

Theorem 2 (Hadwiger, [13]). $\mathrm{Val}_n(V)$ *is one-dimensional, and spanned by a Lebesgue measure.*

We can realize $\mathrm{Val}_n(V)$ as $\mathbf{Dens}\,\boldsymbol{(V)}$, the space of translation-invariant densities on

V. The space $\mathrm{GL}(V)$ of general linear transformations of V acts linearly on $\mathrm{Val}(V)$ by $(g \cdot \phi)(A) := \phi(g^{-1}(A))$. This means $\mathrm{Val}(V)$, and each $\mathrm{Val}_k^{\pm}(V)$, is a representation of $\mathrm{GL}(V)$. Alesker proved in 2011 that:

Theorem 3 (Alesker, [2])**.** *Each* $\mathrm{Val}_k^{\pm}(V)$ *is an irreducible representation of* $\mathrm{GL}(V)$*.*

In Appendix A.2, we state some more sophisticated properties of the representation.

1.2 Smooth valuations

A convex valuation $\phi \in \mathrm{Val}(V)$ is called **smooth** if the Banach-space valued map $\mathrm{GL}(V) \to \mathrm{Val}(V)$ given by $g \mapsto g \cdot \phi$ is smooth. We denote the space of smooth valuations by $\mathbf{Val}^{\infty}(\boldsymbol{V})$. Although this is a subspace of $\mathrm{Val}(V)$, we do not equip it with the subspace topology. Instead, we give $\mathrm{Val}^{\infty}(V)$ the Gårding topology, which is a stronger linear topology than that inherited by $\mathrm{Val}(V)$, and with respect to which $\mathrm{Val}^{\infty}(V)$ is a Fréchet space. This construction of $\mathrm{Val}^{\infty}(V)$, and its Gårding topology, are specific instances of more general objects from representation theory. See Appendix A for more details. McMullen's decomposition and Alesker's irreducibility theorem both apply to $\mathrm{Val}^{\infty}(V)$, and thus we use similar notation for k-homogeneous and even/odd smooth convex valuations.

1.2.1 A characterization of $\mathrm{Val}^{\infty}(V)$

In [4], Section 5, Alesker found an extremely useful characterization of smooth valuations in terms of the normal cycle, which we now describe. First we define the **normal cycle** of $A \in \mathcal{K}(V)$, denoted $\boldsymbol{N(A)}$. At any $x \in A$, the **tangent cone** T_xA and its dual T_x^*A (also called the **normal cone** to A at x) are defined as

$$T_xA := \text{ the closure of } \{y \in V \mid \text{ there exists } \varepsilon > 0 \text{ s.t. } x + \varepsilon y \in A\} \subseteq V$$

$$T_x^*A := \{\xi \in V^* \mid \langle \xi, y\rangle \geq 0 \text{ for all } y \in T_xA\} \subseteq V^*,$$

where V^* denotes the dual space of V. The tangent cone is a closed convex cone in V, and its dual is also closed convex cone in V^*. Then we define the **characteristic cycle** of A, denoted $\boldsymbol{CC(A)}$, by

$$\mathbb{C}(A) := \bigcup_{x \in A} T_x^*(A).$$

The characteristic cycle of A is a closed n dimensional subset of T^*V. Here, we identify T^*V with $V \times V^*$, possible since V is a real vector space. Then, moreover, $\mathbb{C}(A)$ is invariant with respect to multiplication by $\mathbb{R}_{\geq 0}$ on the second factor. Let $\underline{0} := V \times \{0\} \subseteq V \times V^* = T^*V$, and set

$$\underline{CC}(A) := \mathbb{C}(A) \setminus \underline{0}$$
$$\bar{CC}(A) := \underline{CC}(A) / \mathbb{R}_{\geq 0}\,.$$

In this way, $\bar{CC}(A)$ is contained in the **cosphere bundle** $\mathbb{P}_V$ of V, defined by quotienting $T^*V \setminus \underline{0}$ by multiplication of $\mathbb{R}_{\geq 0}$ on the second factor. In other words,

$$\mathbb{P}_V := (T^*V \setminus \underline{0}) / \mathbb{R}_{\geq 0} = V \times \mathbb{P}_+(V^*)$$

where $\mathbb{P}_+(V^*)$ denotes the oriented lines in V^*. Then finally, the normal cycle $N(A)$ of A is taken to be the image of $\bar{CC}(A)$ under the involution on $\mathbb{P}_V$ given by changing the orientation of a line. Now fix an orientation on V. This canonically induces an orientation on $\mathbb{C}(A)$, and hence on $N(A)$.

It can be shown $N(A)$ is closed in $\mathbb{P}_V$. In [7], Section 1.1.12, it is stated $N(A)$ is locally bi-Lipschitz equivalent to $\mathbb{R}^{n-1}$, although no proof of this equivalence is given. Proof notwithstanding, the bi-Lipschitz equivalence allows us to integrate smooth $(n-1)$-forms on $\mathbb{P}_V$ over $N(A)$, and we further get that the resulting linear functional on these forms is continuous.

Let $\Omega^k(\mathbb{P}_V)^{tr}$ denote the set of complex-valued translation-invariant (with respect to translations on V) smooth k-forms on $\mathbb{P}_V$. Then Theorem 5.2.1 in [4] gives a characterization of $\mathrm{Val}^\infty(V)$:

Theorem 4. *For any density $\mu \in \mathrm{Dens}(V)$ and $\omega \in \Omega^{n-1}(\mathbb{P}_V)^{tr}$, the map*

$$\Psi(\mu,\omega) : \mathcal{K}(V) \to \mathbb{R}, \quad A \mapsto \int_A \mu + \int_{N(A)} \omega,$$

is a translation-invariant smooth valuation. Moreover, the map $\Psi : \mathrm{Dens}(V) \oplus \Omega^{n-1}(\mathbb{P}_V)^{tr} \to \mathrm{Val}^\infty(V)$ is linear and surjective.

1.2.2 The product of smooth valuations

In 2004, Alesker in [3] introduced a product on $\mathrm{Val}^\infty(V)$, such that the product map is continuous in the Gårding topology, and associative, commutative, and distributive; thus $\mathrm{Val}^\infty(V)$ equipped with this product is an algebra over $\mathbb{C}$. The unit of this algebra is the Euler characteristic. Furthermore, the product satisfies $\mathrm{Val}_j^\infty(V) \cdot \mathrm{Val}_k^\infty(V) \subseteq \mathrm{Val}_{k+j}^\infty(V)$, and so in light of McMullen's decomposition, $\mathrm{Val}^\infty(V)$ is a graded ring.

In fact, the product map $\mathrm{Val}_k^\infty(V) \times \mathrm{Val}_{n-k}^\infty(V) \to \mathrm{Val}_n^\infty(V)$ is a perfect pairing, in the sense that for any nonzero $\phi \in \mathrm{Val}_k^\infty(V)$, there is $\psi \in \mathrm{Val}_{n-k}^\infty(V)$ such that $\phi \cdot \psi \neq 0$. Equivalently, the induced map

$$\mathrm{Val}_k^\infty(V) \to \mathrm{Val}_{n-k}^\infty(V)^* \otimes \mathrm{Dens}(V)$$
$$\phi \mapsto (\eta \mapsto \phi \cdot \eta)$$

is injective.[2] Equipping the target space $\mathrm{Val}_{n-k}^{\infty}(V)^* \otimes \mathrm{Dens}(V) =: \mathbf{Val}_{\boldsymbol{k}}^{-\boldsymbol{\infty}}\boldsymbol{(V)}$ with the weak-* topology, we then have that $\mathrm{Val}_k^{\infty}(V) \subseteq \mathrm{Val}_k^{-\infty}(V)$ is a dense subspace. We call elements of $\mathrm{Val}_k^{-\infty}(V)$ **generalized translation-invariant convex valuations of degree** $\boldsymbol{k}$, shortening to generalized valuations of degree k when the context is clear. Giving V a Euclidean structure, we can identify $\mathrm{Dens}(V) \cong \mathbb{C}$. This induces an identification $\mathrm{Val}^{\infty}(V)^* \otimes \mathrm{Dens}(V) \cong \mathrm{Val}^{\infty}(V)^*$, which justifies our naming convention. Similarly, we call elements of

$$\mathrm{Val}^{-\infty}(V) := \mathrm{Val}^{\infty}(V)^* \otimes \mathrm{Dens}(V) \text{ and}$$
$$\mathrm{Val}_k^{\pm,-\infty}(V) := \mathrm{Val}_{n-k}^{\pm,\infty}(V)^* \otimes \mathrm{Dens}(V)$$

generalized translation-invariant convex valuations, and generalized even/odd translation-invariant convex valuations of degree k.

The linear action of $\mathrm{GL}(V)$ on $\mathrm{Val}(V)$ induces an action on $\mathrm{Val}^{\infty}(V)$ and $\mathrm{Val}^{-\infty}(V)$. For any subgroup G of $\mathrm{GL}(V)$, we denote the collection of G-invariant valuations by $\mathrm{Val}(V)^G$, and similarly for smooth and generalized translation-invariant k-homogeneous even/odd convex valuations.

1.3 Another View of Smooth and Generalized Valuations

The reference for this Section is [7], Section 1.2.1. In light of Theorem 4 above, let us define a smooth (not necessarily translation-invariant) valuation on a smooth oriented manifold X. Let $\mathcal{P}(X)$ be the set of all submanifolds of X with corners. Let $\mathbb{P}_X$ denote the cosphere bundle on X, defined as in the vector space case. A **smooth valuation** is any map $\mathcal{P}(X) \to \mathbb{C}$ of the form

[2] Here, for a topological vector space E, we let E^* denote the continuous dual of E, with the weak-* topology.

$$A \mapsto \int_A \mu + \int_{N(A)} \omega,$$

where $\mu \in \Omega^n(X)$ and $\omega \in \Omega^{n-1}(\mathbb{P}_X)$, and $N(A)$ is the normal cycle appropriately defined in this setting (see [7], Section 1.2.1, page 22). We denote the collection of smooth valuations by $\boldsymbol{\mathcal{V}^\infty(X)}$. As this space is given as a quotient of $\Omega^n(X) \oplus \Omega^{n-1}(\mathbb{P}_X)$, it can be equipped with a Fréchet topology. A **compactly supported smooth valuation** is a smooth valuation for which there is a compact A in X such that its restriction to $X \smallsetminus A$ is zero; denote the collection of compactly supported smooth valuations by $\mathcal{V}_c^\infty(X)$, carrying the smallest topology such that the imbedding into $\mathcal{V}^\infty(X)$ is continuous. The space $\mathcal{V}_c^\infty(X)$ is the corresponding quotient of $\Omega_c^n(X) \oplus \Omega_c^{n-1}(\mathbb{P}_X)$.

A **generalized valuation** is an element of $\boldsymbol{\mathcal{V}^{-\infty}(X)} := (\mathcal{V}_c^\infty(X))^*$, and is therefore determined by elements of $\mathcal{D}_n(X) \oplus \mathcal{D}_{n-1}(\mathbb{P}_X)$, where $\boldsymbol{\mathcal{D}_k(X)} := \Omega_c^k(X)^*$ is the space of k-dimensional currents on X. We introduced the spaces $\mathcal{V}^\infty(X)$ and $\mathcal{V}^{-\infty}(X)$ because by Proposition 2.5 and preceeding remarks in [10]:

Proposition 5. *In the case $X = V$, we can consider smooth and generalized valuations which are translation-invariant. There is an identification*

$$\mathrm{Val}^\infty(V) \cong \mathcal{V}^\infty(V)^{tr}, \quad \mathrm{Val}^{-\infty}(V) \cong \mathcal{V}^{-\infty}(V)^{tr}.$$

1.4 Preliminary Results

Given a smooth, finite-dimensional vector bundle $\mathcal{E}$ over a manifold M, we denote the space of smooth sections of $\mathcal{E}$ over M by $\Gamma^\infty(M, \mathcal{E})$, and those with compact support by $\Gamma_c^\infty(M, \mathcal{E})$. We define the space of generalized sections of $\mathcal{E}$ as

$$\Gamma^{-\infty}(M,\mathcal{E}) := \Gamma_c^\infty(M,\mathcal{E}^*\otimes|\omega|)^*,$$

where $|\omega|$ is the vector bundle of densities over M, with fiber $\mathrm{Dens}(T_xM)$ over $x\in M$, and $\mathcal{E}^*$ is the dual bundle to $\mathcal{E}$. We give $\Gamma^{-\infty}(M,\mathcal{E})$ the weak-* topology.

We let $\mathrm{Gr}_k(V)$ denote the Grassmanniann of k-dimensional real subspaces of a real vector space V. Often in the sequel, we will take $V=\mathbb{C}^n$, and therefore write $\mathrm{Gr}_k(\mathbb{C}^n)$ for the Grassmanniann of *real* k-planes. Since this symbol normally denotes the complex Grassmanniann, remember this is not the case for us.

1.4.1 The Klain map

Let K^k denote the vector bundle over $\mathrm{Gr}_k(V)$ whose fiber over E is the one-dimensional space $\mathrm{Dens}(E)$. We can construct a map

$$\mathrm{Kl}_k : \mathrm{Val}_k^+(V)\to\Gamma(\mathrm{Gr}_k(V),K^k),$$

called the Klain map, as follows. For $\phi\in\mathrm{Val}_k^+(V)$ and $E\in\mathrm{Gr}_k(V)$, the map $\phi|_E := \phi(E\cap\cdot)$ on $\mathcal{K}(E)$ is an element of $\mathrm{Val}_k(E)$. By Hadwiger's theorem (Theorem 2), it is therefore a multiple of a Lebesgue measure on E, and hence we get $\phi|_E\in\mathrm{Dens}(E)$. Then we set $\mathrm{Kl}_k(\phi)(E) := \phi|_E$. This is a continuous section of K^k, hence Kl_k is well-defined. Klain showed in 2000 that

Theorem 6 (Klain, [14])**.** *The map Kl_k is injective and $\mathrm{GL}(V)$ equivariant.*

Note here that the action of $g\in\mathrm{GL}(V)$ on $s\in\Gamma(\mathrm{Gr}_k(V),K^k)$ is defined by the pull-back:

$$g^*(s)(E) := g_*^{-1} s(gE), \quad E \in \mathrm{Gr}_k(V). \tag{1}$$

Naturally, this definition extends to the case where K^k is replaced by the bundle of l-densities $\mathrm{Dens}(E)^l$, and for dual l-densities $\mathrm{Dens}^*(E)^l = \mathrm{Dens}(E)^{-l}$.

The proof of Klain's theorem relies on his previous result that every simple element of $\mathrm{Val}(V)$ (i.e. one vanishing on sets in $\mathcal{K}(V)$ of dimension less than n) is a multiple of some fixed Lebesgue measure. Klain's theorem is useful because it implies that a valuation $\phi \in \mathrm{Val}_k^+(V)$ is determined by its Klain function $\mathrm{Kl}_k(\phi)$. Note that Klain's map takes smooth valuations to smooth sections, i.e.

$$\mathrm{Kl}_k : \mathrm{Val}_k^{+,\infty}(V) \to \Gamma^\infty(\mathrm{Gr}_k(V), K^k). \tag{2}$$

According to Proposition 4.4 in [6], Klain's map extends uniquely to a $\mathrm{GL}(V)$-equivariant imbedding

$$\mathrm{Kl}_k : \mathrm{Val}_k^{+,-\infty}(V) \to \Gamma^{-\infty}(\mathrm{Gr}_k(V), K^k),$$

which remains injective and has closed image in the weak-* topology.

In the presence of a Euclidean structure on V, we get an associated Lebesgue measure vol on V. This volume induces a Lebesgue measure vol_E on each subspace $E \in \mathrm{Gr}_k(V)$; in the case $V = \mathbb{R}^n$, the induced measures are Hausdorff measures of dimension k. These vol_E let us define a $SO(n)$-invariant continuous global section $s_0 \in \Gamma(\mathrm{Gr}_k(V), K^k)$ by $s_0(E) := \mathrm{vol}_E$, and this section defines a trivialization of K^k. Under this trivialization, any generalized section of Klain's bundle is given by fs_0, where $f \in C^{-\infty}(\mathrm{Gr}_k(V))$ is a generalized function on the Grassmannian.

1.4.2 The Crofton map

Let C^k be the bundle over $\mathrm{Gr}_{n-k}(V)$ whose fiber over E is $\mathrm{Dens}(V/E) \otimes \mathrm{Dens}(T_E \mathrm{Gr}_{n-k}(V))$. Smooth sections of C^k are smooth measures $s \in \mathcal{M}^\infty(\mathrm{Gr}_{n-k}(V), \mathrm{Dens}(V/E))$; indeed, this is the definition of the latter space. We also have the isomorphism

$$\mathcal{M}^\infty(\mathrm{AGr}_{n-k}(V))^{tr} = \mathcal{M}^\infty(\mathrm{Gr}_{n-k}(V), \mathrm{Dens}(V/E)),$$

where the lefthand side is the space of translation-invariant smooth measures on the affine Grassmanniann. We define, for $s \in \Gamma^\infty(\mathrm{Gr}_{n-k}(V), C^k) = \mathcal{M}^\infty(\mathrm{AGr}_{n-k}(V))^{tr}$,

$$\mathrm{Cr}_{n-k}(s)(A) := \int_{\mathrm{AGr}_{n-k}(V)} \chi(A \cap E) ds(E), \quad A \in \mathcal{K}(V).$$

Alesker's irreducibility theorem can be used to show $\mathrm{Cr}_{n-k}(s)$ is a smooth k-homogeneous convex valuation. See [11], Section 2.1, for details. We therefore get a map

$$\Gamma^\infty(\mathrm{Gr}_{n-k}(V), C^k) \to \mathrm{Val}_k^{+,\infty}(V),$$

called the **Crofton map**. Whereas the Klain map was injective,

Theorem 7. *The Crofton map* Cr_{n-k} *is surjective.*

Ultimately, surjectivity is a consequence of Alesker's irreducibility theorem; see Theorem 2.3.2 in [7] for more context. Proposition 4.5 in [6] gives that the Crofton map extends uniquely to a surjection

$$\Gamma^{-\infty}(\mathrm{Gr}_{n-k}(V), C^k) \to \mathrm{Val}_k^{+,-\infty}(V).$$

The Crofton map is also $\mathrm{GL}(V)$-equivariant. We call elements of $\Gamma^{-\infty}(\mathrm{Gr}_{n-k}(V), C^k)$ **generalized Crofton measures of degree $\boldsymbol{k}$** (or Crofton distributions). Crofton distributions are given by sections of a simpler bundle when we are concerned with G-invariants. First, note the well-known fact that the tangent space to the Grassmanniann satisfies

$$T_E \,\mathrm{Gr}_{n-k}(V) = \mathrm{Hom}(E, V/E) = E^* \otimes V/E. \tag{3}$$

Therefore we can write

$$\begin{aligned} \mathrm{Dens}(T_E \,\mathrm{Gr}_{n-k}(V)) &= \mathrm{Dens}(E^* \otimes V/E) && \text{by } 3 \\ &= \mathrm{Dens}^k(E^*) \otimes \mathrm{Dens}^{n-k}(V/E) && \text{distributing over the tensor product} \\ &= \mathrm{Dens}^n(E) \otimes \mathrm{Dens}^{n-k}(V). && (4) \end{aligned}$$

The last line uses the isomorphism $\mathrm{Dens}(V/E) = \mathrm{Dens}(E^*) \otimes \mathrm{Dens}(V)$. We now see the Crofton bundle satisfies

$$C^k|_E = \mathrm{Dens}(V/E) \otimes \mathrm{Dens}(T_E \,\mathrm{Gr}_{n-k}(V)) = \mathrm{Dens}^{n+1}(E^*) \otimes \mathrm{Dens}^{n-k+1}(V).$$

With respect to the action of $\mathrm{GL}(V)$ on the Grassmannian, this identification is $\mathrm{stab}(E)$-equivariant. Since $\mathrm{Dens}^{n-k+1}(V)$ is independent of the fiber E, it follows that G-invariant elements of $\mathrm{Val}_k^{+,-\infty}(V)$ are given by generalized sections $s \in \Gamma^{-\infty}(\mathrm{Gr}_{n-k}(V), \mathrm{Dens}(E^*)^{n+1})^G$.

Now, note that $\mathrm{Dens}(E^*) = \mathrm{Dens}^*(E)$, and also that $(\mathrm{Dens}^l(E))^* = \mathrm{Dens}^{-l}(E)$. Therefore, we view $s \in \Gamma^{-\infty}(\mathrm{Gr}_{n-k}(V), \mathrm{Dens}^{-(n+1)}(E))$.

We can further simplify the situation by considering the Euclidean trivialization as in the previous discussion of Klain's imbedding. In this case, Crofton distributions of degree k are given by fs_0, where $f \in C^{-\infty}(\mathrm{Gr}_{n-k}(V))$ and $s_0 \in \Gamma(\mathrm{Gr}_{n-k}(V), \mathrm{Dens}^{-(n+1)}(E))$ is the continuous global section of the $(n+1)$-dual density bundle on E determined by an auxillary Euclidean structure on V.

1.4.3 The cosine transform

By the action of the Klain map on smooth valuations given by (2), and by the definition of the Crofton map, we can compose the Klain and Crofton maps to get a new map

$$T_{n-k,k} := \mathrm{Kl}_k \circ \mathrm{Cr}_{n-k} : \Gamma^{\infty}(\mathrm{Gr}_{n-k}(V), C^k) \to \Gamma^{\infty}(\mathrm{Gr}_k(V), K^k).$$

We call $T_{n-k,k}$ the **cosine transform**. It is a $\mathrm{GL}(V)$-equivariant map, and its kernel is the Kernel of Cr_{n-k} (since the Klain map is injective).

Now equip V with a Euclidean structure. This induces a volume vol on V, and a volume vol_k on each $E \in \mathrm{Gr}_k(V)$, and thus also trivializes the Klain and Crofton bundles. We can also identify $\mathrm{Gr}_k(V)$ with $\mathrm{Gr}_{n-k}(V)$, and therefore realize the cosine transform as a map $T_k : C^{\infty}(\mathrm{Gr}_k(V)) \to C^{\infty}(\mathrm{Gr}_k(V))$. Define the **cosine of the angle** between $E, F \in \mathrm{Gr}_k(V)$ by

$$|\cos(E,F)| := \frac{\mathrm{vol}_i(\mathrm{Pr}_F(A))}{\mathrm{vol}_i(A)}, \quad A \subseteq E,\ \mathrm{vol}_i(A) > 0,$$

where Pr_F is the orthogonal projection of A onto F. Then the cosine transform T_k is given explicitly by

$$T_k(f)(E) = \int_{\mathrm{Gr}_k(V)} f(F)|\cos(E,F)|dF. \tag{5}$$

In 2004, Alesker and Bernstein in [5] studied the range of this map. According to [9], Section 2.3, the map T_k is self-adjoint, and therefore extends to a map $T_k : C^{-\infty}(\mathrm{Gr}_k(V)) \to C^{-\infty}(\mathrm{Gr}_k(V))$. An extremely useful property of this extension is that on the Dirac delta $\delta_E \in C^{-\infty}(\mathrm{Gr}_k(V))$, we have

$$T_k(\delta_E)(F) = |\cos(E,F)|. \tag{6}$$

This is used in the following lemma:

Lemma 8. *Let $\phi \in \mathrm{Val}_k^{+,\infty}(V)$, and take $m_\phi \in \Gamma^\infty(\mathrm{Gr}_{n-k}(V), C^k)$ such that $\mathrm{Cr}_{n-k}(m_\phi) = \phi$ (possible by surjectivity of the Crofton map). Then, viewing m_ϕ as a smooth measure,*

$$\mathrm{Kl}_k(\phi)(E) = \langle m_\phi, |\cos(E,\cdot)|\rangle, \quad E \in \mathrm{Gr}_k(V).$$

Proof. Simply write

$$\begin{aligned}
\mathrm{Kl}_k(\phi)(E) &= \mathrm{Kl}_k(\mathrm{Cr}_{n-k}(m_\phi))(E) \\
&= \langle T_{n-k,k} m_\phi, \delta_E\rangle && \text{by definition of the Dirac delta} \\
&= \langle m_\phi, T_k \delta_E\rangle && \text{using a Euclidean structure on } V, \text{ and self-adjointness} \\
&= \langle m_\phi, |\cos(E,\cdot)|\rangle && \text{by (6).}
\end{aligned}$$

□

1.4.4 The Alesker-Fourier transform

In 2011, Alesker in [1] found a $\mathrm{GL}(V)$-equivariant linear isomorphism

$$\mathbb{F} : \mathrm{Val}_k^{\pm,\infty}(V) \to \mathrm{Val}_{n-k}^{\pm,\infty}(V^*) \otimes \mathrm{Dens}(V),$$

now called the **Alesker-Fourier transform**. While $\mathbb{F}$ acts on both even and odd valuations, we have a simple characterization of $\mathbb{F}$ only in the even case. For $\phi \in \mathrm{Val}_k^{+,\infty}(V)$, we set $\mathbb{F}\,\phi$ to be the valuation with Klain function

$$\mathrm{Kl}_{\mathbb{F}\,\phi}(E^\perp) := \mathrm{Kl}_\phi(E), \quad E \in \mathrm{Gr}_k(V),$$

where the above is written assuming an underlying Euclidean metric on V. However, we could work in invariant terms, noting that $(E^\perp)^*$ is canonically isomorphic to V/E; in this view, the left side is in

$$\mathrm{Dens}(E^\perp) \otimes \mathrm{Dens}(V) = \mathrm{Dens}^*(V/E) \otimes \mathrm{Dens}(V) = \mathrm{Dens}(E),$$

which is necessary as the right side is also in $\mathrm{Dens}(E)$. In Section 6.2 of [6], the authors find that the Alesker-Fourier transform extends to a $\mathrm{GL}(V)$-equivariant isomorphism $\mathrm{Val}_k^{+,-\infty}(V) \cong \mathrm{Val}_{n-k}^{+,-\infty}(V)$. Therefore, we only need to study the structure of $\mathrm{Val}_k^{+,-\infty}(V)$ for $k \leq [n/2]$.

2 The Indefinite Unitary Group

Consider a complex vector space W, equipped with a hermitian form h of signature (p, q). The **indefinite unitary group** $U(p, q)$, also denoted $U(h)$, is the subgroup of $\mathrm{GL}(W)$ which preserves the form h. Equipping $W \cong \mathbb{C}^n$, $n = p + q$, with a standard basis, h is unitarily equivalent to the sesquilinear form

$$(u, v) \mapsto u_1\overline{v_1} + \cdots + u_p\overline{v_p} - u_{p+1}\overline{v_{p+1}} - \cdots - u_{p+q}\overline{v_{p+q}}.$$

Then, in terms of matrices, $U(p, q)$ is the matrix subgroup of $\mathrm{GL}(n; \mathbb{C})$ such that $M^*\Phi M = \Phi$, where Φ is the diagonal matrix with entry $\Phi_{ii} = 1$ for $i \leq p$ and $\Phi_{ii} = -1$ for $i > p$. The **special indefinite unitary group** $SU(p, q)$ is the subgroup of all $M \in U(p, q)$ with determinant 1.

An immediate consequence of the fact $M^*\Phi M = \Phi$ is:

$$1 = |\det \Phi| = |\det M^*\Phi M| = |\det M|^2,$$

so $|\det M| = 1$ for $M \in U(1, 1)$. Then, since M is invertible, the matrix M^*M is hermitian and positive-definite. So then $|M| := \sqrt{M^*M}$, where we take the positive-definite square root, is well-defined, hermitian, and positive-definite.

2.1 The Group $U(1, 1)$

We now consider the case of $U(1, 1) \leq \mathrm{GL}(n, \mathbb{C})$. Here $\Phi = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. The following proposition and its proof are due to Simon, found in [17], page 566-567.

Proposition 9. *(i) If $M \in U(1, 1)$ and $M \geq 0$, then M has the form*

$$M = \begin{pmatrix} \cosh t & u \sinh t \\ \overline{u} \sinh t & \cosh t \end{pmatrix} =: P(t,u), \tag{7}$$

$$\textit{or equivalentely } M = \rho^{-1} \begin{pmatrix} 1 & -\overline{a} \\ -a & 1 \end{pmatrix} =: A(\alpha),$$

for some $t \geq 0$ *and* $|u| = 1$, *or* $|\alpha| < 1$ *with* $\rho = \sqrt{1 - |\alpha|^2}$.

(ii) Any $M \in U(1,1)$ *can be expressed uniquely as* $M = U|M|$, *where* U *and* $|M|$ *are in* $U(1,1)$, U *is unitary, and* $|M| \geq 0$.

(iii) Any unitary transformation in $U(1,1)$ *has the form* $\left(\begin{smallmatrix} u_1 & 0 \\ 0 & u_2 \end{smallmatrix}\right)$, *for some* $|u_1| = |u_2| = 1$.

(iv) Any $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in U(1,1)$ *satisfies*

$$|a| = |d|, \quad |b| = |c|, \quad |a|^2 - |b|^2 = 1 \tag{8}$$

$$\textit{and } a\overline{b} - c\overline{d} = 0. \tag{9}$$

Conversely any M *satisfying both* (8) *and* (9) *is in* $U(1,1)$. *As a consequence,*

$$SU(1,1) = \left\{ \begin{pmatrix} a & b \\ \overline{b} & \overline{a} \end{pmatrix} : |a|^2 - |b|^2 = 1 \right\}. \tag{10}$$

(v) The space $U(1,1)$ *is homeomorphic to* $S^1 \times S^1 \times \mathbb{D}$ *and* $SU(1,1)$ *is homeomorphic to* $S^1 \times \mathbb{D}$, *where* $\mathbb{D}$ *is the open unit disk in* $\mathbb{C}^2$. *As a corollary, both* $U(1,1)$ *and* $SU(1,1)$ *are connected.*

Proof. (i) Suppose $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in U(1,1)$ is hermitian and $M \geq 0$. Then $\det M$ is real and positive, so $\det M = 1$ and $M \in SU(1,1)$. Therefore $M^{-1} = \left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right)$, and

$$\Phi = M^*\Phi M \iff M = \Phi M^{-1}\Phi = \begin{pmatrix} d & b \\ c & a \end{pmatrix}.$$

Thus $d = a$, and since M is positive-definite, $a > 0$. This means $1 = \det M = a^2 - bc$, and so $bc \in \mathbb{R}$, with neither b nor c zero. Then $c = \rho\bar{b}$ for some $\rho \in \mathbb{R}$, and therefore $1 = a^2 - \rho|b|^2$. On the other hand, condition (8) holding (an easy consequence of the fact $M \in U(1,1)$) means $1 = a^2 - |b|^2$, so $\rho = 1$.

In sum, $a = d > 0$ and $c = \bar{b}$ and $a^2 - |b|^2 = 1$. Therefore $a = \cosh t$ and $|b| = \sinh t$ for some $t \in \mathbb{R}$, and $b = u \sinh t$ for some $|u| < 1$. So then $M = P(t,u)$, as desired. It is easily seen $P(t,u) = A(\alpha)$ for some $|\alpha| < 1$.

(ii) By polar decomposition, we can uniquely write $M = U|M|$, where U is unitary and $|M| = \sqrt{M^*M} \geq 0$. All that remains is to show U and $|M|$ are in $U(1,1)$. Since $M \in U(1,1)$, then $M^*M \in SU(1,1)$ and is positive-definite. Therefore by (i), we have $M^*M = P(t,u)$ for $t \geq 0$ and $|u| = 1$. A straightforward computation then shows

$$|M| = \sqrt{P(t,u)} = P(t/2, u) \in SU(1,1).$$

This also implies $U = M|M|^{-1} \in U(1,1)$.

(iii) If $U \in U(1,1)$ is unitary, then the condition $U^*\Phi U = \Phi$ becomes $\Phi U = U\Phi$, meaning U is diagonal. Its entries having unit norm is then a consequence of the fact $U \in U(1,1)$ implies condition (8) holds.

(iv) Let $M \in U(1,1)$, and write $M = U|M|$. Then by (i) - (iii), we can write $U = \left(\begin{smallmatrix} u_1 & 0 \\ 0 & u_2 \end{smallmatrix}\right)$

and $|M| = P(t, u_3)$. We get

$$M = U|M| = \begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix} \begin{pmatrix} \cosh t & u_3 \sinh t \\ \overline{u_3} \sinh t & \cosh t \end{pmatrix} = \begin{pmatrix} u_1 \cosh t & u_1 u_3 \sinh t \\ u_2 \overline{u_3} \sinh t & u_2 \cosh t \end{pmatrix}.$$

Since we can see any M satisfying (8) and (9) must be of this form, then $M \in U(1,1)$. The converse implication is easily verified, and was already used in parts (i) and (iii). Finally, note $M \in SU(1,1)$ if and only if $u_1 = \overline{u_2}$, which gives description (10) of $SU(1,1)$.

(v) In (iii) we saw that $K := U(2) \cap U(1,1)$ is the group of diagonal unitary matrices. Furthermore, using (ii) and (iii), we see that polar decomposition expresses any $M \in U(1,1)$ uniquely as a product of $U \in K$ and $|M| = A(\alpha) \in SU(1,1)$, for $|\alpha| < 1$. Therefore, the map

$$\begin{aligned} K \times \{A(\alpha) : |\alpha| < 1\} &\to U(1,1) \\ (U, A(\alpha)) &\mapsto UA(\alpha) \end{aligned}$$

is one-to-one and onto. It is straightforward to check this map is also a homeomorphism. Then, since K is homeomorphic to $S^1 \times S^1$ and $\{A(\alpha)\}$ is homeomorphic to $\mathbb{D}$, we get the desired result $U(1,1) \cong S^1 \times S^1 \times \mathbb{D}$. The case $SU(1,1)$ is seen via exactly similar reasoning.

$\square$

2.2 The Action of $U(p,q)$ on $\mathrm{Gr}_k(\mathbb{C}^n)$

Consider $\mathbb{C}^n$ equipped with hermitian inner product h of signature (p,q), given by the matrix Φ as defined above. We equip $\mathbb{R}^{2n}$ with the complex structure $J := \left(\begin{smallmatrix} 0 & -I_n \\ I_n & 0 \end{smallmatrix}\right)$, where I_n is the $n \times n$ identity matrix. Then we identify $(\mathbb{R}^n, J)$ with $\mathbb{C}^n$ by

$$(x_1, \dots, x_{2n}) = (x_1 + ix_{n+1}, x_2 + ix_{n+2}, \dots, x_n + ix_{2n}).$$

We also get an identification $\mathrm{GL}(n,\mathbb{C}) \leq \mathrm{GL}(2n,\mathbb{R})$. In this way, h defines a signature $(2p, 2q)$ non-degenerate quadratic form Q on $\mathbb{R}^{2n}$ defined by

$$Q(x) := h(x,x) = x_1^2 + x_{n+1}^2 + \cdots + x_p^2 + x_{n+p}^2 - x_{p+1}^2 - x_{n+p+1}^2 - \cdots x_n^2 - x_{2n}^2.$$

We also use Q to denote the accompanying bilinear form on $\mathbb{R}^{2n}$, given by $Q(x,y) := \mathrm{Re}\, h(x,y)$. Clearly from the definition of Q, we have $U(p,q) \leq O(2p,2q) = O(Q)$. The **Kähler form** ω on $\mathbb{C}^n$ associated to h is defined by $\omega(u,v) := -\mathrm{Im}\, h(u,v)$. The form ω is a symplectic form on $\mathbb{R}^{2n}$, and in this picture we have the relation $Q(u,v) = \omega(u, Jv)$.

Now consider $V \in \mathrm{Gr}_k(\mathbb{C}^n)$ (recall our convention that this denotes the real Grassmanniann). Then $Q|_V$ has signature (a,b), where $0 \leq a, b$ and $a + b \leq k$. Let

$$\Lambda_{a,b}(k) := \{V \in \mathrm{Gr}_k(\mathbb{C}^n) : Q|_V \text{ has signature } (a,b)\}. \tag{11}$$

Note that according to Section 4.1 of [9], Witt's theorem implies the non-empty $\Lambda_{a,b}(k)$ are the orbits of $O(2p,2q)$ acting on $\mathbb{R}^{2n}$. As the action of $U(p,q)$ on $\mathrm{Gr}_k(\mathbb{C}^n)$ preserves the signature of $Q|_V$ (since it preserves h, hence Q), the orbits of this action are subsets of $\Lambda_{a,b}(k)$. In the non-degenerate case $V \in \Lambda_{a,b}(k)$ where $a + b = k$, we say a basis of

$\{u_1, \ldots, u_k\}$ of V is **Q-orthonormal** if $Q(u_i, u_j) = 0$ for $i \neq j$ and $Q(u_i) \in \{-1, 1\}$ with $Q(u_i) \leq Q(u_j)$ when $i \leq j$. Such a basis always exists.

Since $(\mathbb{C}^n, \omega)$ is a real symplectic vector space, we define for any real vector subspace V,

$$V^\omega := \{u \in \mathbb{C}^n \mid \omega(u, v) = 0 \text{ for all } v \in V\},$$

and say V is **symplectic** if $V \cap V^\omega = \{0\}$, **isotropic** if $V \subseteq V^\omega$, and **Lagrangian** if $V = V^\omega$. Since $U(p, q)$ preserves ω, we may also declare orbits in the Grassmanniann to be symplectic, isotropic, or Lagrangian according to the status of any representative.

We denote by P the usual Euclidean inner product on $\mathbb{R}^{2n}$. The forms P and Q are **compatible**, in the sense that we can decompose $\mathbb{R}^{2n} = V_{2p} \oplus V_{2q}$, where $\dim V_{2p} = 2p$ and $\dim V_{2q} = 2q$, such that

$$V_{2p} = \{x : P(x) = Q(x) \text{ and } V_{2q} = \{x : P(x) = -Q(x)\}.$$

There is an involution $S : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ such that $Q(x, y) = P(x, Sy)$. In coordinates, S is given by changing the sign of the even-indexed coordinates.

2.3 The Differential of the Action

For now, we do not use a complex structure, and work in $\mathbb{R}^n$. For $g \in \mathrm{GL}(n, \mathbb{R})$, define the function $\psi_g : \mathrm{Gr}_k(\mathbb{R}^n) \to \mathbb{R}$ by

$$\psi_g(E) := |\det(\mathrm{Jac}(g : E \to gE))|^{-2},$$

where $\mathrm{Jac}(g : E \to gE)$ denotes the Jacobian matrix of the function $E \to gE$ given by

$x \mapsto gx$. This Jacobian is calculated by equipping E and gE with the Euclidean structure induced by P. Since g is linear as a function $E \to gE$, its Jacobian is constant, so ψ_g is well-defined.

For $E \in \mathrm{Gr}_k(\mathbb{R}^n)$, let $\{u_i\}$ be a P-orthonormal basis. Let $M(E)$ denote the matrix with entry $M(E)_{ij} := Q(u_i, u_j)$. Then let $\tau(E) \in [0, \pi/2]$ be defined by

$$\cos 2\tau(E) := \det M(E).$$

This definition is independent of the choice of P-orthonormal basis. We have from [9], Proposition 4.7:

Proposition 10. *(a) If $Q|_E$ is non-degenerate, and $g \in O(p,q)$, then*

$$\psi_g(E) = \frac{\cos 2\tau(gE)}{\cos 2\tau(E)}.$$

(b) $\psi_g \equiv 1$ for $g \in O(n)$.

(c)

$$|\det \mathrm{Jac}(g : \mathrm{Gr}_k(\mathbb{R}^n) \to \mathrm{Gr}_k(\mathbb{R}^n))|_E| = \psi_g(E)^{n/2} |\det g|^k,$$

where the Jacobian is calculated by fixing any $O(n)$-invariant Riemannian metric on the Grassmanniann.

Proof. (a) Suppose $E \in \mathrm{Gr}_k(\mathbb{R}^n)$ is such that $Q|_E$ is non-degenerate, and let $g \in O(p,q)$. Take a basis $\{v_i\}_{i=1}^k$ of E. By definition,

$$\cos 2\tau(E) = \frac{\det Q(v_i, v_j)}{\det P(v_i, v_j)}, \quad \cos 2\tau(gE) = \frac{\det Q(gv_i, gv_j)}{\det P(gv_i, gv_j)}.$$

As we assume $g \in O(p, q)$, we must have $\det Q(v_i, v_j) = \det Q(gv_i, gv_j)$, which combined with the above gives

$$\frac{\cos 2\tau(gE)}{\cos 2\tau(E)} = \frac{\det P(v_i, v_j)}{\det P(gv_i, gv_j)} = \psi_g(E).$$

(b) This is straightforward.

(c) First, note $\psi_g \equiv 1$ for any $g \in O(n)$. So then, appealing to (b), perhaps by composing with $g \in SO(n)$, we may assume $gE = E$. We need to determine the action of g on $\mathrm{Dens}(T_E \operatorname{Gr}_k(\mathbb{R}^n))$. Recall that (4) gives us

$$\mathrm{Dens}(T_E \operatorname{Gr}_k(\mathbb{R}^n)) = \mathrm{Dens}^n(E^*) \otimes \mathrm{Dens}^k(\mathbb{R}^n).$$

Now, say $\mathrm{Dens}(E) = \mathrm{span}(\mathrm{vol}_E)$, where vol_E is the Lebesgue volume induced by P on E. The action of g on $\mathrm{Dens}(E)$ is given by $g \cdot \mathrm{vol}_E = g_*^{-1} \mathrm{vol}_E$, and therefore satisfies

$$\int_E f \circ g^{-1} \psi_{g^{-1}}(E)^{-1/2} d \,\mathrm{vol}_E = \int_E f d \,\mathrm{vol}_E .$$

Since $\psi_{g^{-1}}(E)$ is constant, the above tells us that $g \cdot \mathrm{vol}_E = \psi_{g^{-1}}(E)^{1/2} \mathrm{vol}_E$. Note here $\psi_{g^{-1}}(E) = \psi_g(E)^{-1}$, so g acts on $\mathrm{Dens}(E)$ by the scalar $\psi_g(E)^{-1/2}$. Then g acts on $\mathrm{Dens}^n(E^*)$ by $\psi_g(E)^{n/2}$. As we can clearly see g acts by the scalar $|\det g|^k$ on

$\mathrm{Dens}^k(\mathbb{R}^n)$, then we conclude g acts on $\mathrm{Dens}(T_E \mathrm{Gr}_k(\mathbb{R}^n)) = \mathrm{Dens}^n(E^*) \otimes \mathrm{Dens}^k(\mathbb{R}^n)$ by the scalar $\psi_g(E)^{n/2}|\det g|^k$, as desired.

□

Now consider the case $(\mathbb{R}^{2n}, J) \cong \mathbb{C}^n$. In practice, when searching for $U(p,q)$-invariant generalized sections of Klain and Crofton's bundles, we use the Euclidean trivialization as defined in Sections 1.4.1 and 1.4.2. In doing so, we get the following lemma

Lemma 11. *Generalized sections of Klain and Crofton's bundles K^k and C^{n-k} invariant under action of $U(p,q)$ correspond to generalized functions $f, g \in C^{-\infty}(\mathrm{Gr}_k(\mathbb{C}^n))$ transforming by*

$$M^* f = \psi_M^{1/2} f, \quad M^* h = \psi_M^{-(n+1)/2} h$$

respectively, for all $M \in U(p,q)$.

Proof. Consider first Klain's bundle. Suppose $s \in \Gamma^{-\infty}(\mathrm{Gr}_k(\mathbb{C}^n), K^k)$ is $U(p,q)$-invariant. Using the Euclidean trivialization, write $s = f s_0$, where $s_0 \in \Gamma(\mathrm{Gr}_k(\mathbb{C}^n), \mathrm{Dens}(E))$ comes from the Euclidean structure P on $\mathbb{C}^n$. Then

$$M^* s = s \iff M^*(f s_0) = f s_0 \iff M^* f M^* s_0 = f s_0 \iff M^* f = f \frac{s_0}{M^* s_0}.$$

Now we can compute, for $E \in \mathrm{Gr}_k(\mathbb{C}^n)$ and $A \subseteq E$ measurable,

$$\begin{aligned} M^* s_0(E)(A) &= (M^{-1})_* s_0(ME)(A) && \text{by definition} \\ &= s_0(ME)(MA) \\ &= \psi_g^{-1/2}(E) s_0(E)(A) && \text{by choice of } s_0. \end{aligned}$$

The claim for Klain's bundle immediately follows.

Now consider Crofton's bundle. As in the end of Section 1.4.2, use the Euclidean trivialization to identify a $U(p,q)$-invariant Crofton distribution of degree $n-k$ with hs_0, where $h \in C^{-\infty}(\mathrm{Gr}_k(\mathbb{C}^n))$ and $s_0 \in \Gamma(\mathrm{Gr}_k(\mathbb{C}^n), \mathrm{Dens}^{-(n+1)}(E))$ comes from the Euclidean structure P on $\mathbb{C}^n$. Then the claim for Crofton's bundle follows in the same way as in Klain's case, from the observation that

$$M^* s_0(E)(A) = \psi_M^{(n+1)/2}(E) s_0(E)(A), \quad E \in \mathrm{Gr}_k(\mathbb{C}^n),\ A \subseteq E.$$

□

3 The Action of $U(1,1)$ on $\mathrm{Gr}_k(\mathbb{C}^2)$

We now investigate in detail the case of $U(1,1)$ acting on $\mathrm{Gr}_k(\mathbb{C}^2)$ for $k=1$ and 2. Let $\{\boldsymbol{e_i}\}_{i=1}^2$ be the standard basis of $\mathbb{C}^2$ as a complex vector space.

3.1 The case $k=1$

For this section, we denote $\Lambda_{a,b}(1) = \Lambda_{a,b}$ (recall definition (11)). Let $\mathbb{P}_+(\mathbb{C}^2)$ denote the real oriented lines in $\mathbb{C}^2$ (that is, equivalence classes of the relation $\left(\begin{smallmatrix} z \\ w \end{smallmatrix}\right) \sim \lambda\left(\begin{smallmatrix} z \\ w \end{smallmatrix}\right)$ for $\lambda > 0$). Then $GL(2,\mathbb{C})$ acts on $\mathbb{P}_+(\mathbb{C}^2)$ by $M \cdot [z:w] := [M\left(\begin{smallmatrix} z \\ w \end{smallmatrix}\right)]$. We claim

Proposition 12. *The orbits of the action of $U(1,1)$ on $\mathbb{P}_+(\mathbb{C}^2)$ are as follows:*

orbit	*open/closed*	*dimension*
$\Lambda_{1,0} = U(1,1).[1:0] = \{[z:w] \mid \lvert z\rvert^2 - \lvert w\rvert^2 > 0\}$	*open*	*3*
$\Lambda_{0,1} = U(1,1).[0:1] = \{[z:w] \mid \lvert z\rvert^2 - \lvert w\rvert^2 < 0\}$	*open*	*3*
$\Lambda_{0,0} = U(1,1).[1:1] = \{[z:w] \mid \lvert z\rvert^2 - \lvert w\rvert^2 = 0\}$	*closed*	*2*

The first two columns would be the exact same if we were to consider the action of $U(1,1)$ on unoriented lines $\mathbb{P}(\mathbb{C}^2) = \mathrm{Gr}_1(\mathbb{C}^2)$, or on complex lines $\mathbb{P}^{\mathbb{C}}(\mathbb{C}^2)$. In the former case, the dimensions also agree, but in the latter, they are different.

To compute the dimension of the closed orbit, we need the following result, Theorem 21.18 from [15]:

Theorem 13. *Suppose G is a Lie group and M is a G-homogeneous space (that is, M is a smooth manifold on which G acts smoothly and transitively). Then for any $p \in M$, the stabilizer G_p is closed, and the map $G/G_p \to M$ defined by $gG_p \mapsto g \cdot p$ is an equivariant diffeomorphism.*

Remark 14. *Here we put the canonical smooth structure on the coset space G/G_p. It has dimension $\dim G - \dim G_p$ (see [15] Theorem 21.17).*

Proof of the proposition. Consider $[z : w] \in \mathbb{P}_+(\mathbb{C}^2)$. For $|z|^2 - |w|^2 > 0$ (resp. < 0) [resp. $= 0$], define

$$M := \frac{1}{\sqrt{|z|^2 - |w|^2}} \begin{pmatrix} z & \overline{w} \\ w & \overline{z} \end{pmatrix}$$
$$\left(\text{resp. } M := \frac{1}{\sqrt{|w|^2 - |z|^2}} \begin{pmatrix} \overline{w} & z \\ \overline{z} & w \end{pmatrix} \right)$$
$$\left[\text{resp. } M := \frac{1}{|z|} \begin{pmatrix} z & 0 \\ 0 & w \end{pmatrix} \right].$$

By Proposition 9, $M \in U(1,1)$. Then noting that $[z : w] = M \cdot [1 : 0]$ (resp. $M \cdot [0 : 1]$) [resp. $M \cdot [1 : 1]$] completes the characterization of the orbits. To see the orbits are distinct, note that $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \cdot [1 : 0] = [0 : 1]$ implies $a = 0$, which is impossible for $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in U(1,1)$. Similarly, if $M \cdot [1 : 1] = [1 : 0]$ (resp. $[0 : 1]$) then $a = b$ (resp. $c = d$), which is again impossible for indefinite unitary M.

To compute the dimensions of the orbits, notice that the first two $\Lambda_{1,0}$ and $\Lambda_{0,1}$ are open in $\mathbb{P}_+(\mathbb{C}^2)$, and therefore must have full dimension 3. Fort the last orbit, notice we realized $\Lambda_{0,0}$ as the orbit of $[1:1]$ under the action of the subgroup $U(1,1) \cap U(2)$. As $U(1,1).[1:1] = (U(1,1) \cap U(2)).[1:1]$ is closed, it is a submanifold of $\mathbb{P}_+(\mathbb{C}^2)$. This means it is a $(U(1,1) \cap U(2))$-homogeneous space, and so by Theorem 13, its dimension is equal to $\dim U(1,1) \cap U(2) - \dim(U(1,1) \cap U(2))_{[1:1]}$. We find that $\left(\begin{smallmatrix} u_1 & 0 \\ 0 & u_2 \end{smallmatrix}\right) \cdot [1:1] = [1:1]$ if and only if $u_1 = u_2 \in \mathbb{R}_{>0}$, so that $(U(1,1) \cap U(2))_{[1:1]} = \{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)\}$. This has dimension 0, so $\dim \Lambda_{0,0} = 2 - 0 = 2$. $\square$

Note that since all elements of $\mathbb{P}(\mathbb{C}^2)$ are one-dimensional, the three orbits are isotropic.

3.2 The case $k = 2$

Here denote $\Lambda_{a,b}(2) = \Lambda_{a,b}$.

Theorem 15. *The orbits of the action of $U(1,1)$ on $\mathrm{Gr}_2(\mathbb{C}^2)$ are characterized by the following table:*

Signature	*Orbit representatives*	*Dimension*	*sym/iso/Lag*
$\Lambda_{2,0}$	$V_{2,0}^{\theta} := \mathrm{span}_{\mathbb{R}}\left(\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} i\cosh\theta \\ \sinh\theta \end{smallmatrix}\right)\right)$, $\theta \geq 0$	*2 if* $\theta = 0$, *3 else*	*sym*
$\Lambda_{0,2}$	$V_{0,2}^{\theta} := \mathrm{span}_{\mathbb{R}}\left(\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} \sinh\theta \\ i\cosh\theta \end{smallmatrix}\right)\right)$, $\theta \geq 0$	*2 if* $\theta = 0$, *3 else*	*sym*
$\Lambda_{1,1}$	$V_{1,1}^{\theta} := \mathrm{span}_{\mathbb{R}}\left(\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} i\sinh\theta \\ \cosh\theta \end{smallmatrix}\right)\right)$, $\theta \geq 0$	*3*	*Lag if* $\theta = 0$, *sym else*
$\Lambda_{1,0}$	$\mathrm{span}_{\mathbb{R}}\left(\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} i \\ 1 \end{smallmatrix}\right)\right)$	*3*	*sym*
$\Lambda_{0,1}$	$\mathrm{span}_{\mathbb{R}}\left(\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 \\ i \end{smallmatrix}\right)\right)$	*3*	*sym*
$\Lambda_{0,0}$	$V_{0,0}^{\pm} := \mathrm{span}_{\mathbb{R}}\left(\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} i \\ \pm i \end{smallmatrix}\right)\right)$	*1*	*Lag for* $V_{0,0}^{+}$, *sym for* $V_{0,0}^{-}$.

In light of this description, when $a + b = 2$ let $\Lambda_{a,b}^{\theta} := U(1,1).V_{a,b}^{\theta}$ for $\theta \geq 0$, and also let $\Lambda_{0,0}^{\pm} := U(1,1).V_{0,0}^{\pm}$. The remainder of this subsection is devoted to proving Theorem 15. First, we justify the orbit representatives; then we compute the dimensions; and finally we justify whether the orbits are symplectic, isotropic, or Lagrangian.

Remark 16. *The parametrization of the non-degenerate orbits by a single real parameter is in analogy with the case of $U(n)$ acting on $\mathrm{Gr}_k(\mathbb{C}^n)$. Here, the orbits are parametrized by the multiple Kähler angle $\boldsymbol{\theta}$, defined by Tasaki in [18]. For $k \leq n$, the angle $\boldsymbol{\theta}$ is a tuple $(\theta_1, \ldots, \theta_{[k/2]})$, with $\theta_i \in [0, \pi/2]$. In the case $k = 2$, the multiple Kähler angle satisfies $\cos\boldsymbol{\theta} = \omega(u, v)$, where ω is the Kähler form on $\mathbb{C}^n$ determined by the usual Euclidan structure.*

3.2.1 Classifying the orbit representatives

Useful $U(1,1)$ transformations

We list some useful transformations. Denote by $G(u', v')$ a linear transformation mapping u' to v'. Under the indicated conditions, the following transformations are in $U(1,1)$:

- $G(u, \boldsymbol{e_1}) := \left(\begin{smallmatrix} u_1 & \overline{u_2} \\ u_2 & \overline{u_1} \end{smallmatrix}\right)^{-1}$, when $Q(u, u) = 1$. Note this is in $SU(1,1)$.
- $G\left(x, \left(\begin{smallmatrix} x_1 \\ |x_2| \end{smallmatrix}\right)\right) := \left(\begin{smallmatrix} 1 & 0 \\ 0 & \frac{\overline{x_2}}{|x_2|} \end{smallmatrix}\right)$, when $x_2 \neq 0$. Moreover, this fixes $\boldsymbol{e_1}$. Note this is in $\{\left(\begin{smallmatrix} 1 & 0 \\ 0 & d \end{smallmatrix}\right) \mid |d| = 1\}$, which we can identify with S^1.
- $G(u, \boldsymbol{e_2}) := \left(\begin{smallmatrix} u_2 & \overline{u_1} \\ u_1 & \overline{u_2} \end{smallmatrix}\right)^{-1}$, when $Q(u, u) = -1$. Note this is in $SU(1,1)$.
- $G\left(x, \left(\begin{smallmatrix} |x_1| \\ x_2 \end{smallmatrix}\right)\right) := \left(\begin{smallmatrix} \frac{\overline{x_1}}{|x_1|} & 0 \\ 0 & 1 \end{smallmatrix}\right)$, when $x_1 \neq 0$. Moreover, this fixes $\boldsymbol{e_2}$. Note this is in $\{\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right) \mid |a| = 1\}$, which we can identify with S^1.
- $G(u, \boldsymbol{e_1} + \boldsymbol{e_2}) := \left(\begin{smallmatrix} u_1 & 0 \\ 0 & u_2 \end{smallmatrix}\right)^{-1}$, when $|u_1| = |u_2| = 1$. Note this is in $\{\left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right) \mid |a| = |d| = 1\}$, which we can identify with $S^1 \times S^1$.

The case $\Lambda_{a,b}$ with $a + b = 2$

Suppose $Q|_V$ is non-degenerate. Consider any Q-orthonormal basis $\{u', v'\}$ of V. Then $\{-u', v'\}$ is also a Q-orthonormal basis of V. Since $\omega(-u', v') = -\omega(u', v')$, we can therefore always take a Q-orthonormal basis $\{u, v\}$ of V such that $\omega(u, v) \geq 0$.

Lemma 17. *If $Q|_V$ is non-degenerate with signature $(2,0)$ or $(0,2)$, then a Q-orthonormal basis $\{u,v\}$ such that $\omega(u,v) \geq 0$ further satisfies $\omega(u,v) \geq 1$.*

Proof. Suppose $Q|_V$ has signature $(2,0)$. Then $Q(u) = Q(v) = 1$, so $G(u, \boldsymbol{e_1}) \in U(1,1)$ takes $u \mapsto \boldsymbol{e_1}$ and $v \mapsto G(u, \boldsymbol{e_1})v =: x$. Since $U(1,1)$ preserves h, it preserves $\omega(u,v)$, and therefore $0 \leq \omega(u,v) = -\operatorname{Im} h(\boldsymbol{e_1}, x) = \operatorname{Im}(x_1)$. Moreover, since $U(1,1)$ preserves Q, we find $0 = Q(u,v) = Q(\boldsymbol{e_1}, x)$, and so $\operatorname{Re}(x_1) = 0$. Thus $x_1 = i\omega(u,v)$, and therefore $1 = Q(v) = Q(x) = \omega(u,v)^2 - |x_2|^2 = 1$. So then $\omega(u,v)^2 \geq 1$, and as $\omega(u,v) \geq 0$, we conclude $\omega(u,v) \geq 1$.

Now supposing $Q|_V$ has signature $(0,2)$. Then $Q(u) = Q(v) = -1$, and so $G(u, \boldsymbol{e_2}) \in U(1,1)$ takes $u \mapsto \boldsymbol{e_2}$ and $v \mapsto G(u, \boldsymbol{e_2})v =: x$. By similar arguments to those in the previous case, since $U(1,1)$ preserves both ω and Q, we find $x_2 = i\omega(u,v)$. Then $-1 = Q(x) = |x_1|^2 - \omega(u,v)^2$, and so $\omega(u,v)^2 \geq 1$. Since $\omega(u,v) \geq 0$, we again conclude $\omega(u,v) \geq 1$. □

In light of this lemma, given V such that $Q|_V$ is non-degenerate with signature $(2,0)$ or $(0,2)$ we may define the **Kähler angle** $\theta := \cosh^{-1} \omega(u,v) \geq 0$, where $\{u,v\}$ is a Q-orthonormal basis of V such that $\omega(u,v) \geq 1$. Note this is independent of the choice of basis. Clearly θ is invariant under action by $U(1,1)$.

Proposition 18. *Suppose $Q|_V$ is non-degenerate with signature $(2,0)$ or $(0,2)$ respectively, and has Kähler angle $\theta \geq 0$. Then V is in the orbit of*

$$\operatorname{span}_{\mathbb{R}}\left(\boldsymbol{e_1}, \left(\begin{smallmatrix} i\cosh\theta \\ \sinh\theta \end{smallmatrix}\right)\right) \text{ or } \operatorname{span}_{\mathbb{R}}\left(\boldsymbol{e_2}, \left(\begin{smallmatrix} \sinh\theta \\ i\cosh\theta \end{smallmatrix}\right)\right), \text{ respectively.}$$

In particular, the orbits are completely characterized by the signature of $Q|_V$ and θ.

Proof. Suppose $Q|_V$ has signature $(2,0)$. By the proof of Lemma 17, we already know V to be in the orbit of $\operatorname{span}_{\mathbb{R}}(\boldsymbol{e_1}, x)$, where $x_1 = i\cosh\theta$ and $|x_2| = \sqrt{|x_1|^2 - 1} = \sinh\theta$. If $\theta = 0$, then we are done since $|x_2| = 0$ means $x_2 = 0$. Otherwise, acting on $\operatorname{span}_{\mathbb{R}}(\boldsymbol{e_1}, x)$ by $G\left(x, \left(\begin{smallmatrix} x_1 \\ |x_2| \end{smallmatrix}\right)\right) \in U(1,1)$ shows V is in the desired orbit.

If $Q|_V$ has signature $(0,2)$, then Lemma 17 shows V is in the orbit of $\mathrm{span}_{\mathbb{R}}(\boldsymbol{e_2}, x)$, where $x_2 = i\cosh\theta$ and $|x_1| = \sinh\theta$. If $\theta = 0$, again we are done. Otherwise act by $G\left(x, \left(\begin{smallmatrix}|x_1|\\ x_2\end{smallmatrix}\right)\right)$. $\square$

Now we consider the case where $Q|_V$ has signature $(1,1)$. Here, we define the Kähler angle to be $\theta := \sinh^{-1}\omega(u,v) \geq 0$, where $\{u,v\}$ is a Q-orthonormal basis of V such that $\omega(u,v) \geq 0$. Again, this is invariant under change of basis.

Proposition 19. *Suppose $Q|_V$ has signature $(1,1)$ and V has Kähler angle $\theta \geq 0$. Then V is in the orbit of $\mathrm{span}_{\mathbb{R}}\left(\boldsymbol{e_1}, \left(\begin{smallmatrix}i\sinh\theta\\ \cosh\theta\end{smallmatrix}\right)\right)$. In particular, the orbits are completely characterized by θ.*

Proof. By choice of $\{u,v\}$, we have $Q(u) = 1$ and $Q(v) = -1$. Then $G(u, \boldsymbol{e_1}) \in U(1,1)$, and it takes $u \mapsto \boldsymbol{e_1}$ and $v \mapsto G(u,\boldsymbol{e_1})v =: x$. Exactly as in Lemma 17, we find that $U(1,1)$ preserving ω and Q implies $x_1 = i\omega(u,v)$. Then $-1 = Q(x) = \omega(u,v)^2 - |x_2|^2$, and so $|x_2| = \cosh\theta$. Then, since $|x_2| > 0$, we can act on $\mathrm{span}_{\mathbb{R}}(\boldsymbol{e_1}, x)$ by $G\left(x, \left(\begin{smallmatrix}x_1\\ |x_2|\end{smallmatrix}\right)\right) \in U(1,1)$ to see V is in the desired orbit. $\square$

Remark 20. *Note that, for $a = 0, 2$, we have realized $\Lambda^0_{a,b}$ as the orbits of $SU(1,1)$ acting on $\mathrm{Gr}_2(\mathbb{C}^2)$.*

The degenerate case $\Lambda_{a,b}$ with $a + b = 1$

Proposition 21. *Suppose $Q|_V$ has signature $(1,0)$ or $(0,1)$, respectively. Then V is in the orbit of $\mathrm{span}_{\mathbb{R}}\left(\left(\begin{smallmatrix}1\\ 0\end{smallmatrix}\right), \left(\begin{smallmatrix}i\\ 1\end{smallmatrix}\right)\right)$ or $\mathrm{span}_{\mathbb{R}}\left(\left(\begin{smallmatrix}0\\ 1\end{smallmatrix}\right), \left(\begin{smallmatrix}1\\ i\end{smallmatrix}\right)\right)$, respectively.*

Proof. Suppose $Q|_V$ has signature $(1,0)$. We get a basis $\{u,v\}$ of V with $Q(u) = 1$, $Q(u,v) = Q(v) = 0$, and $\omega(u,v) \geq 0$. Then $G(u,\boldsymbol{e_1}) \in U(1,1)$ takes $u \mapsto \boldsymbol{e_1}$ and $v \mapsto G(u,\boldsymbol{e_1})v =: x$. Since $U(1,1)$ preserves ω and h, the same argument as in Lemma 17 gives that $x = i\omega(u,v)$. Then $0 = Q(x,x) = \omega(u,v)^2 - |x_2|^2$, and so $\omega(u,v) = |x_2|$. Now, if $|x_2| = 0$, then $x_2 = x_1 = 0$, which is impossible since V has real dimension 2. Therefore, we can act on $\mathrm{span}_{\mathbb{R}}(\boldsymbol{e_1}, x)$

by $G\left(x, \left(\begin{smallmatrix} x_1 \\ |x_2| \end{smallmatrix}\right)\right) \in U(1,1)$ to see that V is in the orbit of $\operatorname{span}_{\mathbb{R}}(\boldsymbol{e_1}, \omega(u,v)(i\boldsymbol{e_1} + \boldsymbol{e_2})) = \operatorname{span}_{\mathbb{R}}(e_1, i\boldsymbol{e_1} + \boldsymbol{e_2})$.

If $Q|_V$ has signature $(0,1)$, then emulate the previous case, acting first by $G(u, \boldsymbol{e_2})$ and then by $G\left(x, \left(\begin{smallmatrix} |x_1| \\ x_2 \end{smallmatrix}\right)\right) \in U(1,1)$. $\square$

The degenate case $\Lambda_{0,0}$

Proposition 22. *Suppose $Q|_V \equiv 0$. Then V in the orbit of either $V_{0,0}^+ = \operatorname{span}_{\mathbb{R}}\left(\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} i \\ i \end{smallmatrix}\right)\right)$ or $V_{0,0}^- := \operatorname{span}_{\mathbb{R}}\left(\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} i \\ -i \end{smallmatrix}\right)\right)$.*

Proof. We can find a basis $\{u, v\}$ of V such that $Q(u) = Q(v) = Q(u,v) = 0$, $\omega(u,v) \geq 0$ (perhaps taking $-u$), and also $|u_1| = |u_2| = 1$ (the first equality is $Q(u) = 0$, and the second is achieved by scaling). Then $G(u, \boldsymbol{e_1} + \boldsymbol{e_2}) \in U(1,1)$ takes $u \mapsto \boldsymbol{e_1} + \boldsymbol{e_2}$ and $v \mapsto G(u, \boldsymbol{e_1} + \boldsymbol{e_2}) =: x$, so V is in the orbit of $\operatorname{span}_{\mathbb{R}}(\boldsymbol{e_1} + \boldsymbol{e_2}, x)$.

Moreover, since $U(1,1)$ preserves Q, we get that $0 = Q(u,v) = Q(\boldsymbol{e_1} + \boldsymbol{e_2}, x) = \operatorname{Re}(x_1) - \operatorname{Re}(x_2)$. So then $0 = Q(v) = Q(x)$ implies $|\operatorname{Im}(x_1)| = |\operatorname{Im}(x_2)|$, i.e. $\operatorname{Im}(x_2) = \pm \operatorname{Im}(x_1)$. Therefore, V is in the orbit of either

$$
\begin{aligned}
&\operatorname{span}_{\mathbb{R}}\left(\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} x_1 \\ x_1 \end{smallmatrix}\right)\right) = \operatorname{span}_{\mathbb{R}}\left(\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right), \operatorname{Re}(x_1)\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right) + \operatorname{Im}(x_1)\left(\begin{smallmatrix} i \\ i \end{smallmatrix}\right)\right) = \operatorname{span}_{\mathbb{R}}\left(\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} i \\ i \end{smallmatrix}\right)\right) \\
\text{or }\ &\operatorname{span}_{\mathbb{R}}\left(\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} x_1 \\ \overline{x_1} \end{smallmatrix}\right)\right) = \operatorname{span}_{\mathbb{R}}\left(\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right), \operatorname{Re}(x_1)\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right) + \operatorname{Im}(x_1)\left(\begin{smallmatrix} i \\ -i \end{smallmatrix}\right)\right) = \operatorname{span}_{\mathbb{R}}\left(\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} i \\ -i \end{smallmatrix}\right)\right).
\end{aligned}
$$

Note that the fact V (and $G(u, \boldsymbol{e_1} + \boldsymbol{e_2}) \cdot V$) has real dimension 2 ensures that $\operatorname{Im}(x_1) \neq 0$, so the final equalities hold. It remains to show that the orbits are distinct. For a contradiction, suppose that $G \in U(1,1)$ is such that $G \cdot V_{0,0}^+ = V_0^-$. Then $x := G(\boldsymbol{e_1} + \boldsymbol{e_2})$ and ix are in $V_{0,0}^-$. However, all elements of $V_{0,0}^-$ are of the form $\left(\begin{smallmatrix} y_i \\ \overline{y_i} \end{smallmatrix}\right)$, so then $x \in V_{0,0}^-$ implies $x_2 = \overline{x_1}$ and $ix \in V_{0,0}^-$ therefore implies $i = -i$, which is false. We conclude the orbits are in fact distinct. $\square$

Remark 23. *Notice that in this proof, we realized $\Lambda_{0,0}^{\pm}$ as orbits of the action of the matrix* *subgroup $U(2) \cap U(1,1)$, namely $\{\left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right) : |a| = |d| = 1\}$. Since this subgroup is compact, its action on $\mathrm{Gr}_2(\mathbb{C}^2)$ is proper. Therefore, its orbits $\Lambda^{\pm}_{0,0}$ are compact, hence closed (see [15], corollary 21.8).*

3.2.2 Dimension computations

The dimension of the orbits $\Lambda_{a,b}$ with $a+b=1$ are 3, a fact which is shown in Proposition 4.2 of [9]. It was also shown there that $\Lambda_{a,b}$ with $a = 0, 2$ are open subsets of $\mathrm{Gr}_2(\mathbb{C}^2)$, and are therefore smooth (sub)manifolds of dimension 4. Since the map $\Lambda_{a,b} \to \mathbb{R}$ given by $E \mapsto \theta$, where θ is the Kähler angle of E, is continuous, the orbits $\Lambda^{\theta}_{a,b}$ are closed in $\Lambda_{a,b}$.

Therefore, we can use Lee's theorem 13. In our application, we take $G := U(1,1) \cap U(2)$ (resp. $G := U(1,1)$) [resp. $G := SU(1,1)$], and M to be the orbits $\Lambda^{\pm}_{0,0}$ (resp. $\Lambda^{\theta}_{a,b}$, $a = 0, 2$ and $\theta > 0$, and $\Lambda^{\theta}_{1,1}$) [resp. $\Lambda^{0}_{a,b}$, $a = 0, 2$]. Since the orbits M are closed they are submanifolds of $\mathrm{Gr}_2(\mathbb{C}^2)$ ([resp. $\Lambda_{a,b}$, $a = 0, 2$]), and are in particular G-homogeneous spaces. So then by Theorem 13, each orbit M is diffeomorphic to G/G_p, where $p \in M$ is the representative distinguished in Theorem 15. In particular, the dimension of M is equal to $\dim G - \dim G_p$.

- First consider the orbits $\Lambda^{\pm}_{0,0}$. We have already realized these as the orbits of the smooth action of $U(1,1) \cap U(2) \cong S^1 \times S^1$. We compute the stabilizer subgroups to be

$$
\begin{aligned}
(U(1,1) \cap U(2))_{V^{+}_{0,0}} &= \left\{ \begin{pmatrix} u_1 & 0 \\ 0 & u_1 \end{pmatrix} : |u_1| = 1 \right\}, \\
\text{and } (U(1,1) \cap U(2))_{V^{-}_{(0,0)}} &= \left\{ \begin{pmatrix} u_1 & 0 \\ 0 & \overline{u_1} \end{pmatrix} : |u_1| = 1 \right\},
\end{aligned}
$$

  which have dimension 1. Therefore $\dim \Lambda^{\pm}_{0,0} = 2 - 1 = 1$.

Remark 24. *Note that in $U(1,1)$, the stabilizers of $\Lambda^{\pm}_{0,0}$ both contain all matrices*

$\left(\begin{smallmatrix} a & b \\ b & a \end{smallmatrix}\right)$, *with* $a, b \in \mathbb{R}$ *and* $a^2 - b^2 = 1$. *Therefore neither of* $\Lambda^{\pm}_{0,0}$ *have compact stabilizer.*

- Now consider $\Lambda^0_{a,b}$, with $a = 0, 2$. These are the orbits of the action of $SU(1,1)$, and in both cases we observe the stabilizer satisfies

$$SU(1,1)_{V^0_{a,b}} = \left\{ \begin{pmatrix} u & 0 \\ 0 & \overline{u} \end{pmatrix} : |u| = 1 \right\} = SU(1,1) \cap U(2),$$

  which has dimension 1. Therefore $\dim \Lambda^0_{a,b} = 3 - 1 = 2$.

- Third, consider $\Lambda^0_{1,1}$. We can compute the stabilizer of $V^0_{1,1}$ in $U(1,1)$ is

$$U(1,1)_{V^0_{1,1}} = \left\{ \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} : \alpha, \beta, \gamma, \delta \in \mathbb{R},\ |\alpha| = |\delta|,\ |\beta| = |\gamma|,\ \text{and } |\alpha|^2 - |\beta|^2 = 1 \right\},$$

  which has dimension 1. Therefore $\dim \Lambda^0_{1,1} = 4 - 1 = 3$.

Finally, we compute the dimensions of the orbits of the rest of the $\Lambda^\theta_{a,b}$ via the following proposition:

Proposition 25. *The stabilizers of* $V^\theta_{a,b}$, $\theta > 0$ *in* $U(1,1)$ *are*

representative	*stabilizer*	*dimension*	*compact?*
$V^\theta_{2,0}$	$\left\{ \left(\begin{smallmatrix} \alpha + i\beta\cosh\theta & \beta\sinh\theta \\ \beta\sinh\theta & \alpha - i\beta\cosh\theta \end{smallmatrix}\right) : \alpha^2 + \beta^2 = 1 \right\}$	*1*	*Yes*
$V^\theta_{0,2}$	*the conjugate of the case for* $V^\theta_{(0,2)}$, $\theta > 0$	*1*	*Yes*
$V^\theta_{1,1}$	$\left\{ \left(\begin{smallmatrix} \alpha + i\beta\sinh\theta & \beta\cosh\theta \\ \beta\cosh\theta & \alpha - i\beta\sinh\theta \end{smallmatrix}\right) : \alpha^2 - \beta^2 = 1 \right\}$	*1*	*No*

Proof. The arguments in each case are similar, so we only present the case of $V^\theta_{2,0}$, $\theta > 0$. Suppose $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in U(1,1)_{V^\theta_{2,0}}$. Then the condition $M \cdot V^\theta_{2,0} = V^\theta_{2,0}$ is equivalent to the existence of real α, β, γ and δ such that

$$a = \alpha + i\beta \cosh\theta \tag{12}$$

$$c = \beta \sinh\theta \tag{13}$$

$$ia\cosh\theta + b\sinh\theta = \gamma + i\delta\cosh\theta \tag{14}$$

$$ic\cosh\theta + d\sinh\theta = \delta\sinh\theta. \tag{15}$$

First, using (12) and (13) and the fact $|a|^2 - |c|^2 = 1$, we get $\alpha^2 + \beta^2 = 1$. Now substitute (13) into (15), cancel the resulting $\sinh\theta$ (here we use $\theta > 0$)[3] and solve for d to get $d = \delta - i\beta\cosh\theta$. Then $|a| = |d|$ implies $\delta = \pm\alpha$. We claim $\delta = \alpha$. First, note if $\alpha = 0$, then $\delta = \alpha = 0$ is immediate.

Now for a contradiction, suppose $0 \neq \alpha = -\delta$. Then $a = -d$, and the condition $\overline{a}b - cd = 0$ (from our characterization of $U(1,1)$, with the fact c is real) implies $\overline{a}b = -ca$. Using this, we have

$$\begin{aligned}
i\overline{a}a\cosh\theta - ca\sinh\theta &= \overline{a}(\gamma + i\delta\cosh\theta) && \text{multiplying (14) by } \overline{a}\\
i\overline{a}\cosh\theta - c\sinh\theta &= \frac{\overline{a}}{a}(\gamma + i\delta\cosh\theta) && \text{since } M \in U(1,1) \text{ implies } a \neq 0\\
\frac{i\alpha\cosh\theta + \beta\cosh^2\theta - \beta\sinh^2\theta}{\gamma + i\delta\cosh\theta} &= \frac{\overline{a}}{a} && \text{using (12) and (13)}\\
\frac{\beta + i\alpha\cosh\theta}{\gamma - i\alpha\cosh\theta} &= \frac{\alpha - i\beta\cosh\theta}{\alpha + i\beta\cosh\theta} && \text{since } \alpha = -\delta
\end{aligned}$$

Now, as the right side has norm 1, so does the left side. Thus $\gamma = \pm\beta$. If $\gamma = \beta$, then cross-multiplying and comparing the imaginary parts of each side gives $1 = \alpha^2 + \beta^2 = -(\alpha^2 + \beta^2) = -1$, a contradiction[4]. On the other hand, if $\gamma = -\beta$, then the left side is -1. This means $\alpha = 0$ which contradicts our assumption.

[3] in the case $V^{\theta}_{1,1}$, we cancel $\cosh\theta$, so we do not yet use $\theta > 0$

[4] in the case $V^{\theta}_{1,1}$, we get $1 = \alpha^2 - \beta^2 = -\alpha^2 + \beta^2 = -1$, and it is here we need $\theta > 0$

So $\delta = \alpha$. Then $d = \overline{a}$, and the condition $\overline{a}b - cd = 0$ implies $b = c$. Therefore necessarily

$$M = \begin{pmatrix} \alpha + i\beta\cosh\theta & \beta\sinh\theta \\ \beta\sinh\theta & \alpha - i\beta\cosh\theta \end{pmatrix},$$

where $\alpha^2 + \beta^2 = 1$. Conversely, it is easily seen any M of this form stabilizes $V^\theta_{2,0}$, so we are done. $\square$

In light of this proposition, for the remaining $\Lambda^\theta_{a,b}$ we have $\dim \Lambda^\theta_{a,b} = 4 - 1 = 3$.

Finally, the classifications of the orbits as symplectic, isotropic, or Lagrangian is easy to verify working in the respective bases.

4 Non-Constructive Results on $\mathrm{Val}^{-\infty}(\mathbb{C}^2)^{U(1,1)}$

The object of study for the remainder of this work is the space $\mathrm{Val}^{-\infty}(\mathbb{C}^2)^{U(1,1)}$ of $U(1,1)$-invariant translation-invariant generalized valuations. Our first important result is

Theorem 26. *The space of $U(1,1)$ generalized translation-invariant valuations on $\mathbb{C}^2$ is finite-dimensional.*

The proof is based on the following representation-theoretical result from Aizenbud, Gourevitch, and Minchenko (AGM) (see Appendices A and B for the relevant definitions):

Theorem 27 (Aizenbud-Gourevitch-Minchenko, [8])**.** *Let G be a real reductive group, and H a Zariski closed subgroup with Lie algebra $\mathfrak{h}$. If H is a spherical subgroup, then the space $(F^*)^{\mathfrak{h}}$ is finite-dimensional for any admissible irreducible Fréchet representation $(\pi, G(\mathbb{R}), F)$ of $G(\mathbb{R})$ of moderate growth.*

This theorem appears as a corollary to Theorem E in [8], which in turn is derived from their main result, Theorem D.

Proof. (of Theorem 26) AGM's result above immediately implies that $\mathrm{Val}_k^{\pm,-\infty}(\mathbb{C}^2)^{U(1,1)}$ is finite-dimensional provided we can take $G = \mathrm{GL}(\mathbb{C}^2)$, $H = U(1,1)$, and $(\pi, \mathrm{GL}(\mathbb{C}^2), F)$ to be the representation of $\mathrm{GL}(\mathbb{C}^2)$ by $\mathrm{Val}_k^{\pm,\infty}(\mathbb{C}^2)$. So we check these assignments satisfy the hypotheses.

First, it is clear $\mathrm{GL}(\mathbb{C}^2)$ is a real reductive group, and that $U(1,1)$ is a Zariski closed subgroup. To see $U(1,1)$ is spherical, first appeal to Lemma 45 to see that the subgroup B of upper-triangular matrices in $\mathrm{GL}(\mathbb{C}^2)$ is a Borel subgroup. Using the fact $\mathrm{GL}(\mathbb{C}^2)$ acts smoothly and transitively on the space of complex lines $\mathbb{P}^{\mathbb{C}}(\mathbb{C}^2)$, and the stabilizer of $[1:0]$ is precisely B, Theorem 13 gives an equivariant diffeomorphism $\mathrm{GL}(\mathbb{C}^2)/B \cong \mathbb{P}^{\mathbb{C}}(\mathbb{C}^2)$. Since in Proposition 12, we saw $U(1,1)$ acts on $\mathbb{P}^{\mathbb{C}}(\mathbb{C}^2)$ with finitely many orbits, we conclude it is a spherical subgroup of $\mathrm{GL}(\mathbb{C}^2)$.

Second, $\mathrm{Val}_k^{\pm,\infty}(\mathbb{C}^2)$ is shown to be an admissible irreducible Fréchet representation of $\mathrm{GL}(\mathbb{C}^2)$ of moderate growth in Appendix A.2. Hence we may apply Theorem 27 to conclude that $\mathrm{Val}_k^{\pm,-\infty}(\mathbb{C}^2)^{U(1,1)}$ is finite-dimensional. McMullen's decomposition lets us make the same conclusion for $\mathrm{Val}^{-\infty}(\mathbb{C}^2)^{U(1,1)}$.

□

In light of Theorem 26, each $\mathrm{Val}_k^{-\infty}(\mathbb{C}^2)^{U(1,1)}$ for $k \in \{0,1,2,3,4\}$ is finite-dimensional. As we stated Section 1.1,

- $\mathrm{Val}_0(\mathbb{C}^2)$ is one-dimensional and spanned by the Euler characteristic.
- Hadwiger showed $\mathrm{Val}_4(\mathbb{C}^2)$ is one-dimensional and spanned by a Lebesgue measure.

Therefore, it is easy to see $\mathrm{Val}_k^{-\infty}(\mathbb{C}^2)^{U(1,1)}$ is one-dimensional for $k = 0$ and 4. Now, since multiplication by -1 is in $U(1,1)$, any $U(1,1)$-invariant generalized valuation is necessarily even. In other words, $\mathrm{Val}_k^{-\infty}(\mathbb{C}^2)^{U(1,1)} = \mathrm{Val}_k^{+,-\infty}(\mathbb{C}^2)^{U(1,1)}$. This fact allows us to use Alesker's Fourier transform (Section 1.4.4) to get an isomorphism

$$\mathrm{Val}_1^{-\infty}(\mathbb{C}^2)^{U(1,1)} \cong \mathrm{Val}_3^{-\infty}(\mathbb{C}^2)^{U(1,1)}.$$

Therefore $\dim \mathrm{Val}_1^{-\infty}(\mathbb{C}^2)^{U(1,1)} = \dim \mathrm{Val}_3^{-\infty}(\mathbb{C}^2)^{U(1,1)}$. This leaves only two cases untreated, namely

$$\mathrm{Val}_1^{+,-\infty}(\mathbb{C}^2)^{U(1,1)} \text{ and } \mathrm{Val}_2^{+,-\infty}(\mathbb{C}^2)^{U(1,1)}.$$

5 The Space $\mathrm{Val}_1^{-\infty}(\mathbb{C}^2)^{U(1,1)}$

Our goal is to show $\mathrm{Val}_1^{-\infty}(\mathbb{C}^2)^{U(1,1)}$ is 2-dimensional. First, note that any $U(1,1)$-invariant generalized valuation is necessarily $O(2,2)$-invariant, when we view $\mathbb{C}^2$ as $\mathbb{R}^4$. In [9], Bernig and Faifman characterized (among others) the space $\mathrm{Val}_3^{-\infty}(\mathbb{R}^4)^{O(2,2)} = \mathrm{Val}_3^{+,-\infty}(\mathbb{R}^4)^{O(2,2)}$. Since we have the Alesker-Fourier transform, such a characterization also describes the 1-homogeneous case. While their result is in Section 5 of [9], and uses the theory of currents, we use their work to give another view of the characterization via the Klain map. First, we take their Proposition 4.3:

Proposition 28 (Bernig-Faifman)**.** *The dimension of the space of $O(p,q)$-invariant generalized sections of Klain's bundle K^k equals the number of open $O(p,q)$-orbits in $\mathrm{Gr}_k(\mathbb{R}^n)$. A basis of $\Gamma^{-\infty}(\mathrm{Gr}_k(\mathbb{R}^n), K^k)^{O(p,q)}$ is given by the continuous sections κ_a defined by*

$$\kappa_a(E)(v_1 \wedge \cdots \wedge v_k) := \begin{cases} |\det(Q(v_i, v_j))_{i,j=1}^k|^{1/2} & E = \mathrm{span}(v_1, \ldots, v_k) \in \Lambda_{a,k-a}(k) \\ 0 & \textit{otherwise} \end{cases},$$

where $\max(0, k-q) \le a \le \min(k,p)$*, and Q is the non-degenerate bilinear form corre-*

sponding to $O(p,q)$.

In particular, $\Gamma^{-\infty}(\mathrm{Gr}_3(\mathbb{R}^4), K^3)^{O(2,2)} = \mathrm{span}(\kappa_1, \kappa_2)$. Since the Klain map gives an injection

$$\mathrm{Val}_3^{+,-\infty}(\mathbb{R}^4)^{O(2,2)} \to \Gamma^{-\infty}(\mathrm{Gr}_3(\mathbb{R}^4), K^3)^{O(2,2)},$$

we find $\dim \mathrm{Val}_3^{+,-\infty}(\mathbb{R}^4)^{O(2,2)} \leq 2$. Bernig and Faifman then gave two $O(2,2)$-invariant generalized even 3-homogeneous valuations $\phi_{1,0}$ and $\phi_{0,1}$ whose Klain functions are $-\frac{8}{3}\pi\kappa_1$ and $-\frac{8}{3}\pi\kappa_2$ (see [9], Proposition 8.1; note $\phi_{1,0}$ and $\phi_{0,1}$ were defined via invariant Crofton distributions). Therefore, the dimension of the space of $O(2,2)$-invariant generalized 1 or 3-homogeneous valuations on $\mathbb{R}^4$ is exactly 2.

In the $U(1,1)$-invariant case, the above implies $\dim \mathrm{Val}_1^{+,\infty}(\mathbb{C}^2)^{U(1,1)} \leq 2$. We show it is exactly 2, which means any $U(1,1)$-invariant generalized valuation is also $O(2,2)$-invariant. First, we need Lemma A.1 from [9]:

Lemma 29. *Let G be a group acting on a manifold X equipped with a G-equivariant vector bundle $\mathcal{E}$, and let Z be a closed orbit of G. For $\alpha \geq 0$, let F^α be the bundle over Z defined by*

$$F^\alpha|_E := \mathrm{Sym}^\alpha(N_E Z) \otimes \mathrm{Dens}^*(N_E Z) \otimes \mathcal{E}|_E, \quad E \in Z.$$

If $\dim \Gamma^\infty(Z, F^\alpha)^G = 0$ for all $\alpha \geq 0$, then $\dim \Gamma_Z^{-\infty}(X, \mathcal{E})^G = 0$.

We do not prove this lemma here, but note that according to Bernig and Faifman, it is a well-known result. Using this lemma, we now prove

Proposition 30. *There are no non-trivial $U(1,1)$-invariant generalized sections of Klain's bundle supported on the closed orbit $Z = \Lambda_{0,0}(1)$.*

Proof. Note we have

$$F^{\alpha}|_E := \mathrm{Sym}^{\alpha}(N_E Z) \otimes \mathrm{Dens}^*(N_E Z) \otimes \mathrm{Dens}(E), \quad E \in Z.$$

In light of Lemma 29, it suffices to show $(F^{\alpha}|_E)^{U(1,1)_E} = 0$, where $U(1,1)_E := \mathrm{stab}(E) \subseteq U(1,1)$.

First we describe the normal space $N_E Z$ for $E \in \Lambda_{0,0}(1)$. Since Q is degernerate on E, we have $E = E_0 := E \cap E^Q$, where E^Q is the Q-orthogonal complement of E. Then according to Proposition 4.2 of [9], there is an $O(2,2)$ (hence $U(1,1)$)-equivariant isomorphism $N_E Z = N_E \Lambda_{0,0}(1) = \mathrm{Sym}^2 E^*$.

Now we find an element of $U(1,1)_E$ that acts on E by multiplication by nonzero $\lambda \neq \pm 1$. Since $N_E Z = \mathrm{Sym}^2 E^*$, any such element acts on $F^{\alpha}|_E$ by $\lambda^{-2\alpha}\lambda^{-2}\lambda^{-1} \neq 1$, which implies $(F^{\alpha}|_E)^{U(1,1)_E} = 0$ as desired. Take

$$M_\lambda := \begin{pmatrix} \alpha & \gamma \\ \gamma & \alpha \end{pmatrix} \in U(1,1)$$

where (α, γ) is the unique solution to $\alpha + \gamma = \lambda$ and $\alpha^2 - \gamma^2 = 1$. Then in fact $M_\lambda \in U(1,1)_{[1:1]}$, and M_λ acts on elements of $[1:1]$ as multiplication by λ. Therefore, writing $E = g \cdot [1:1]$, we have $g_\lambda := M_\lambda g^{-1} \in U(1,1)_E$ acting on E as multiplication by λ. This completes the proof. □

Finally, we use Proposition 30 to get our main result, which is that

Theorem 31. *The space of $U(1,1)$-invariant 1-homogeneous generalized valuations is 2-dimensional, and any such valuation is also $O(2,2)$-invariant.*

Proof. Recall we have an injection

$$\mathrm{Val}_1^{+,-\infty}(\mathbb{C}^2)^{U(1,1)} \to \Gamma^{-\infty}(\mathrm{Gr}_1(\mathbb{C}^2), K^1)^{U(1,1)}, \tag{16}$$

so that the dimension of the range bounds above that of the domain. Since the Klain bundle is one-dimensional, so is the space of $U(1,1)$-invariant generalized sections supported over the open orbits $\Lambda_{0,1}(1)$ and $\Lambda_{1,0}(1)$, respectively (that these are the open orbits is due to Proposition 12). As Proposition 28 already gives the section κ_0 over $\Lambda_{0,1}(1)$ and κ_1 over $\Lambda_{1,0}(1)$, if we have some $s \in \Gamma^{-\infty}(\mathrm{Gr}_1(\mathbb{C}^2), K^1)^{U(1,1)}$ independent of $\{\kappa_0, \kappa_1\}$, then we must necessarily get an invariant section supported on $\Lambda_{0,0}(1)$. But by Proposition 30, there is no such section. Therefore, we conclude $\Gamma^{-\infty}(\mathrm{Gr}_1(\mathbb{C}^2), K^1)^{U(1,1)} = \mathrm{span}(\kappa_0, \kappa_1)$, and in particular the dimension of $U(1,1)$-invariant generalized sections of Klain's bundle is 2.

This means $\dim \mathrm{Val}_1^{+,-\infty}(\mathbb{C}^2)^{U(1,1)} \leq 2$. Since as we discussed earlier, Bernig and Faifman showed $\dim \mathrm{Val}_1^{+,-\infty}(\mathbb{R}^4)^{O(2,2)} = 2$, and we have $\mathrm{Val}_1^{+,-\infty}(\mathbb{R}^4)^{O(2,2)} \subseteq \mathrm{Val}_1^{+,-\infty}(\mathbb{C}^2)^{U(1,1)}$, we conclude these spaces are the same. Noting any $U(1,1)$-invariant generalized valuation is necessarily even completes the proof. □

By using the Alesker-Fourier transform, we also have $\dim \mathrm{Val}_3^{-\infty}(\mathbb{C}^2)^{U(1,1)} = 2$, and any $U(1,1)$-invariant 3-homogeneous generalized valuation is necessarily $O(2,2)$-invariant.

6 The Cover of $\mathrm{Gr}_2(\mathbb{C}^2)$ by $S^2 \times S^2$

Our next goal is to describe the space $\mathrm{Val}_2^{-\infty}(\mathbb{C}^2)^{U(1,1)}$. As opposed to the 1-homogeneous case, where we used the Klain map, we will instead use the Crofton map

$$\Gamma^{-\infty}(\mathrm{Gr}_2(\mathbb{C}^2), C^2)^{U(1,1)} \to \mathrm{Val}_2^{+,-\infty}(\mathbb{C}^2)^{U(1,1)} = \mathrm{Val}_2^{-\infty}(\mathbb{C}^2)^{U(1,1)}.$$

Since the Crofton map is surjective, to find $U(1,1)$-invariant 2-homogeneous generalized valuations, we will first find $U(1,1)$-invariant Crofton distributions. Since these are generalized sections of a bundle over $\mathrm{Gr}_2(\mathbb{C}^2)$, we will use the convenient fact that $\mathrm{Gr}_2(\mathbb{C}^2)$ is double covered by $S^2 \times S^2$.

6.1 Defining the Cover

We describe how $S^2 \times S^2$ (viewed inside $\mathbb{R}^3 \times \mathbb{R}^3$) covers $\mathrm{Gr}_2(\mathbb{C}^2)$. Let $\{e_i\}_{i=1}^4$ denote the standard basis of $\mathbb{R}^4$, and set the usual Euclidean structure P on $\mathbb{C}^2$. For $(z,w) \in S^2 \times S^2$, let

$$\begin{array}{lll} x_{12} := \dfrac{w_1 + z_1}{2} & x_{34} := \dfrac{w_1 - z_1}{2} & x_{13} := -\dfrac{w_2 + z_2}{2} \\ x_{24} := \dfrac{w_2 - z_2}{2} & x_{14} := \dfrac{w_3 + z_3}{2} & x_{23} := \dfrac{w_3 - z_3}{2}. \end{array}$$

Then set

$$\tau := \sum_{1 \le i < j \le 4} x_{ij} e_i \wedge e_j.$$

We say that (z,w) **generates** τ. This is a simple unit 2-vector, and so defines a 2-plane $E := \{v \in \mathbb{R}^4 \mid v \wedge \tau = 0\}$ in $\mathbb{R}^4$. Conversely, given $E \in \mathrm{Gr}_2(\mathbb{C}^2)$, take a P-orthonormal basis $\{u,v\}$. Then $\tau := u \wedge v \in \bigwedge^2 \mathbb{R}^4$ is a simple unit 2-vector generated from some $(z,w) \in S^2 \times S^2$, and this (z,w) corresponds to E in the double cover. We also will say τ corresponds to E.

Any bilinear form A on $\mathbb{R}^4$ extends to one on $\bigwedge^2 \mathbb{R}^4$ by

$$A(u \wedge v, x \wedge y) := \left| \begin{pmatrix} A(u,x) & A(u,y) \\ A(v,x) & A(v,y) \end{pmatrix} \right|.$$

Therefore, the relevant forms P, Q, and ω all extend to $\bigwedge^2 \mathbb{R}^4$. In particular, note that

$$Q(e_1 \wedge e_3) = Q(e_2 \wedge e_4) = 1$$
$$Q(e_1 \wedge e_2) = Q(e_1 \wedge e_4) = Q(e_2 \wedge e_3) = Q(e_3 \wedge e_4) = -1.$$

6.2 The Action of $U(1,1)$ on $S^2 \times S^2$

Using the complex structure J on $\mathbb{R}^4$, we can view $\mathbb{C}^2 \cong \mathbb{R}^4$ and also $\mathrm{GL}(\mathbb{C}^2) \leq \mathrm{GL}(\mathbb{R}^4)$. Under this identification, the action of $M \in \mathrm{GL}(\mathbb{R}^4)$ on $E \in \mathrm{Gr}_2(\mathbb{C}^2)$ satisfies

$$\begin{aligned} M \cdot E &:= \{Mv \in \mathbb{R}^4 \mid v \wedge \tau = 0\} \\ &= \{u \in \mathbb{R}^4 \mid M^{-1}u \wedge \tau = 0\} \\ &= \{u \in \mathbb{R}^4 \mid u \wedge M\tau = 0\}. \end{aligned}$$

Note the action of M on τ is given by

$$M\tau := \sum_{k<j} \left(\sum_{i<j} x_{ij} M^{kl,ij} \right) e_k \wedge e_l,$$

where $M^{kl,ij}$ is the determinant of the 2×2 minor of M including rows k and l and columns i and j.

Suppose (z, w) generates the unit 2-vector τ and corresponds to the 2-plane E. We define $M \cdot (z, w)$ to be $(z', w') \in S^2 \times S^2$, where (z', w') generates E. Of course, this is only well-defined up to sign. In the case $M \in O(4)$, the fact M preserves P means $M\tau$ itself is a unit 2-vector. Therefore, by expanding $M\tau = \sum_{i<j} x'_{ij} e_i \wedge e_j$, we can reconstruct (z', w') explicitly in terms of the x'_{ij}, and hence the M and (z, w), by solving the linear system. Otherwise, $M\tau$ is not a unit 2-vector. In this case, to find $M \cdot (z, w)$, we must first take a P-orthonormal basis $\{u', v'\}$ of $M \cdot E$, and then extract (z', w') from the unit 2-vector $\tau' := u' \wedge v'$. Beginning with a Q-orthonormal basis $\{u, v\}$ of $M \cdot E$, we can always extract $\{u', v'\}$ using Gram-Schmidt orthonormalization.

According to the polar decomposition in Proposition 9, we can write any $M \in U(1,1)$ as

$$M = \begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix} \cdot \frac{1}{\sqrt{1 - |\alpha|^2}} \begin{pmatrix} 1 & -\overline{\alpha} \\ -\alpha & 1 \end{pmatrix}, \tag{17}$$

for $|u_1| = |u_2| = 1$ and $|\alpha| < 1$. Note here that $K := U(1) \times U(1) = U(2) \cap U(1,1)$ is the maximal compact subgroup of $U(1,1)$. Therefore we have $U(1,1) = KH$, where H is the collection of matrices of the form $A = \frac{1}{\sqrt{1-|\alpha|^2}} \left(\begin{smallmatrix} 1 & -\overline{\alpha} \\ -\alpha & 1 \end{smallmatrix}\right)$ for $|\alpha| < 1$. Since $K \leq U(2) \leq O(4)$, the above discussion allows us to compute that the action of U on (z, w) satisfies $U \cdot (z, w) = (z', w')$, where

$$z' = \begin{pmatrix} z_1 \operatorname{Re}(u_1 u_2) - z_3 \operatorname{Im}(u_1 u_2) \\ z_2 \\ z_1 \operatorname{Im}(u_1 u_2) + z_3 \operatorname{Re}(u_1 u_2) \end{pmatrix}, \quad w' = \begin{pmatrix} w_1 \operatorname{Re}(\overline{u_1} u_2) - w_3 \operatorname{Im}(\overline{u_1} u_2) \\ w_2 \\ w_1 \operatorname{Im}(\overline{u_1} u_2) + w_3 \operatorname{Re}(\overline{u_1} u_2) \end{pmatrix}. \tag{18}$$

We see that U rotates both the z and w copies of S^2 independently around the z_2 and w_2 axes; orbits of K are of the form $S^2 \times S^2 \cap \{z_2 = c_1, w_2 = c_2\}$ for constants $c_1, c_2 \in [-1, 1]$. Furthermore, multiplication by i satisfies $i \cdot (z, w) = ((-z_1, z_2, -z_3), w)$, so the complex planes are covered by $(0, \pm 1, 0) \times S^2$.

6.3 The Function $\cos 2\tau$

In Section 2.3, we defined the function $\tau : \mathrm{Gr}_k(\mathbb{R}^n) \to [0, \pi/2]$ by $\cos 2\tau(E) := \det M(E)$, where $M(E)$ was the matrix with entry $M(E)_{ij} := Q(u_i, u_j)$, where $\{u_i\}$ is a P-orthornormal basis of E. The following lemma describes the lift of $\cos 2\tau$ to $S^2 \times S^2$.

Lemma 32. *Suppose $(z, w) \in S^2 \times S^2$ sits over the 2-plane $E \in \mathrm{Gr}_2(\mathbb{C}^2)$ with P-orthonormal basis $\{u, v\}$. Then*

$$\cos 2\tau(E) = z_2^2 + w_2^2 - 1. \tag{19}$$

Proof. This is a computation:

$$\begin{aligned}
\cos 2\tau(E) &= \left| \begin{pmatrix} Q(u,u) & Q(u,v) \\ Q(v,u) & Q(v,v) \end{pmatrix} \right| && \text{by definition} \\
&= Q(u \wedge v, u \wedge v) \\
&= x_{13}^2 + x_{24}^2 - x_{12}^2 - x_{14}^2 - x_{23}^2 - x_{34}^2 \\
&= z_2^2 + w_2^2 - 1.
\end{aligned}$$

□

6.4 The Kähler angle

Consider a non-degenerate plane $E \in \mathrm{Gr}_2(\mathbb{C}^2)$. The definition of its Kähler angle θ is contingent on the signature of $Q|_E$. In any case, however, θ is uniquely determined by $\omega(u, v)^2$, where $\{u, v\}$ is a Q-orthonormal basis of E. We want to describe $\omega(u, v)^2$ (and hence θ) in terms of the point $(z, w) \in S^2 \times S^2$ over E.

Lemma 33. *Let (z, w) cover the 2-plane E with Q-orthonormal basis $\{u, v\}$. Then*

$$\omega(u,v)^2 = P(u,u)P(\tilde{v},\tilde{v})w_2^2, \tag{20}$$

where $\tilde{v} := v - \frac{P(u,v)}{P(u,u)}u$.

Proof. Using the Gram-Schmidt process, let

$$\tilde{u} := u \text{ and } \tilde{v} := v - \frac{P(u,v)}{P(u,u)}u$$
$$u' := \frac{\tilde{u}}{\sqrt{P(\tilde{u},\tilde{u})}} \text{ and } v' := \frac{\tilde{v}}{\sqrt{P(\tilde{v},\tilde{v})}}.$$

Then $\{u', v'\}$ is a P-orthonormal basis for E, and

$$\tau = u' \wedge v' = \sum_{i<j} x_{ij} e_i \wedge e_j = \frac{1}{\sqrt{P(u,u)P(\tilde{v},\tilde{v})}} u \wedge v$$

is the unit 2-vector generated by (z,w) and corresponding to E. Since ω is a sympleptic form, we have

$$\omega(\tilde{u},\tilde{v}) = \omega(u,v), \text{ which implies } P(u,u)P(\tilde{v},\tilde{v})\omega(u',v')^2 = \omega(u,v)^2.$$

Now compute

$$
\begin{aligned}
\omega(u',v')^2 &= \left|\begin{pmatrix} 0 & \omega(u',v') \\ \omega(v',u') & 0 \end{pmatrix}\right| && \text{since } \omega \text{ is symplectic} \\
&= \left|\begin{pmatrix} 0 & -Q(u',Jv') \\ -Q(v',Ju') & 0 \end{pmatrix}\right| && \text{since } Q(u,Jv) = \omega(u,v) \\
&= -Q(u' \wedge v', Ju' \wedge Jv') && \text{by definition of } Q \text{ on } \bigwedge^2 \mathbb{R}^4 \\
&= \sum_{\substack{i<j \\ k<l}} x_{ij} x_{kl} Q(e_i \wedge e_j, Je_k \wedge Je_l) && \text{expanding in the basis} \\
&= -x_{12}x_{34} + x_{13}^2 - x_{14}x_{23} - x_{23}x_{14} + x_{24}^2 - x_{34}x_{12} && \text{evaluating} \\
&= w_2^2 && \text{by the definition of } x_{ij}
\end{aligned}
$$

Therefore $\omega(u,v)^2 = P(u,u)P(\tilde{v},\tilde{v})w_2^2$, as desired. □

In the case of planes of signature $(1,1)$, recall the Kähler angle is defined by $\sinh\theta = \omega(u,v)$, where $\theta \geq 0$ and u, v are Q-orthonormal such that $\omega(u,v) \geq 0$. So then in particular, planes E from $\Lambda^0_{1,1}$ necessarily corrospond to $(z,w) \in S^2 \times S^2$ with $w_2 = 0$. This is useful in our next exercise, which is realizing the Lagrangian planes in $\mathrm{Gr}_2(\mathbb{C}^2)$ as a subset of $S^2 \times S^2$.

6.5 The Lagrangian planes

The set of Lagrangian planes $\mathscr{L}$ in $\mathrm{Gr}_2(\mathbb{C}^2)$ is the union $\mathscr{L} = \Lambda^+_{0,0} \cup \Lambda^0_{1,1}$. Now, $\Lambda^+_{0,0}$ has representative $V^+_{0,0}$, which we can compute lifts to $\left(\left(\begin{smallmatrix} 0 \\ \pm 1 \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 \\ 0 \\ \mp 1 \end{smallmatrix}\right)\right)$ in $S^2 \times S^2$. Then, since the orbit $\Lambda^+_{0,0}$ was given as the orbit of the action of $K \leq U(1,1)$, we can use 18 (i.e. the fact K independently rotates each copy of S^2 about the second axis) to find that $\Lambda^+_{0,0}$ lifts as

$$\Lambda^+_{0,0} \to \{(z,w) \in S^2 \times S^2 \mid |z_2| = 1, \text{ and } w_2 = 0\}.$$

As we saw in the previous section, the other Lagrangian orbit $\Lambda^0_{1,1}$ contains only those non-degenerate planes corresponding to $(z,w) \in S^2 \times S^2$ with $w_2 = 0$. Since the case $|z_2| = 1$ corresponds to the degenerate orbit, we expect to find that

Proposition 34. *The orbit $\Lambda^0_{1,1}$ lifts as*

$$\Lambda^0_{1,1} \to \{(z,w) \in S^2 \times S^2 : |z_2| \neq 1, \text{ and } w_2 = 0\}.$$

Proof. We prove this directly. The orbit has representative $V^0_{1,1} = \operatorname{span}_{\mathbb{R}}(\boldsymbol{e_1}, \boldsymbol{e_2})$. Then

$$A \cdot V^0_{1,1} = \operatorname{span}_{\mathbb{R}}\left(\left(\begin{smallmatrix} 1 \\ -\alpha \end{smallmatrix} \right), \left(\begin{smallmatrix} -\overline{\alpha} \\ 1 \end{smallmatrix} \right) \right).$$

Using the Gram-Schmidt process, a P-orthonormal basis for $A \cdot V^0_{1,1}$ is

$$\begin{aligned} \{u', v'\} &= \left\{ \frac{\tilde{u}}{\sqrt{P(\tilde{u},\tilde{u})}}, \frac{\tilde{v}}{\sqrt{P(\tilde{v},\tilde{v})}} \right\} \\ &= \left\{ \frac{1}{\sqrt{1+|\alpha|^2}} \begin{pmatrix} 1 \\ -\alpha \end{pmatrix}, \frac{1}{\sqrt{(1+|\alpha|^2)(1+\lambda^2) - 4\lambda \operatorname{Re}\alpha}} \begin{pmatrix} \lambda - \overline{\alpha} \\ 1 - \lambda\alpha \end{pmatrix} \right\}, \end{aligned}$$

where $\lambda := \frac{2 \operatorname{Re}\alpha}{1+|\alpha|^2}$. First, note that

$$P(\tilde{u},\tilde{u}) P(\tilde{v},\tilde{v}) = (1+|\alpha|^2)^2 - 4(\operatorname{Re}\alpha)^2.$$

We find that the corresponding τ' to $A \cdot V^0_{1,1}$ using the orthonormal basis $\{u', v'\}$ is

$$\begin{aligned}\sqrt{P(\tilde{u},\tilde{u})P(\tilde{v},\tilde{v})}\tau' = v_1 \wedge v_2 = {} & (1-(\operatorname{Re}\alpha)^2)e_1 \wedge e_2 + (\operatorname{Im}\alpha)^2 e_3 \wedge e_4 \\ & + \operatorname{Im}\alpha e_1 \wedge e_3 + \operatorname{Im}\alpha e_2 \wedge e_4 \\ & + (-\operatorname{Re}\alpha \operatorname{Im}\alpha)e_1 \wedge e_4 + (-\operatorname{Re}\alpha \operatorname{Im}\alpha)e_2 \wedge e_3.\end{aligned}$$

Then

$$z' = \frac{1}{\sqrt{P(\tilde{u},\tilde{u})P(\tilde{v},\tilde{v})}}\begin{pmatrix}1-|\alpha|^2 \\ -2\operatorname{Im}\alpha \\ 0\end{pmatrix}$$
$$w' = \frac{1}{\sqrt{P(\tilde{u},\tilde{u})P(\tilde{v},\tilde{v})}}\begin{pmatrix}1-(\operatorname{Re}\alpha)^2+(\operatorname{Im}\alpha)^2 \\ 0 \\ -2\operatorname{Re}\alpha\operatorname{Im}\alpha\end{pmatrix}.$$

As expected, $(z', w') \in S^2 \times S^2$, so $A \cdot V^0_{1,1} = (z', w')$. Since the action of U on any (z, w) simply rotates both spheres independently around the z_2 and w_2 coordinate, we conclude the orbit $\Lambda^0_{1,1}$ lifts to

$$\Lambda^0_{1,1} = \{(z,w) \in S^2 \times S^2 : |z_2| \neq 1, \text{ and } w_2 = 0\}.$$

$\square$

Therefore, the set of Lagrangian planes is $\mathscr{L} = \Lambda^+_{0,0} \cup \Lambda^0_{1,1} = \{(z,w) \in S^2 \times S^2 : w_2 = 0\}$.

7 A Lower Bound for the Dimension of $\mathrm{Val}_2^{-\infty}(\mathbb{C}^2)^{U(1,1)}$

7.1 The Indefinite Orthogonal Case

Our first step to bounding $\dim \mathrm{Val}_2^{-\infty}(\mathbb{C}^2)^{U(1,1)}$ below is to view $\mathrm{Val}_2^{-\infty}(\mathbb{R}^4)^{O(2,2)}$ as inside $\mathrm{Val}_2^{-\infty}(\mathbb{C}^2)^{U(1,1)}$. As in the 1-homogeneous case, Bernig and Faifman in [9] characterized the former space. In Section 5 of [9], using the theory of currents, they proved their Theorem 2, which implies $\dim \mathrm{Val}_2^{-\infty}(\mathbb{C}^2)^{O(2,2)} = 2$. They then defined several Crofton distributions. For $\mathrm{Re}(\lambda) > 0$, and (a,b) with a,b non-negative and $a+b=2$, define the generalized function $m_{a,b}^{(\lambda)} \in C^{-\infty}(\mathrm{Gr}_2(\mathbb{C}^2))$ by

$$\varphi \mapsto \int_{\Lambda_{a,b}(2)} \varphi(E) |\cos 2\tau(E)|^{\lambda} dE.$$

Bernig and Faifman showed, in their Proposition 8.5, that

Proposition 35. *(a) Each $m_{a,b}^{(\lambda)}$ extends meromorphically to $\mathbb{C}$. If $a = 0,2$, then $m_{a,b}^{(\lambda)}$ has simple poles at $\lambda = -m/2$ for $m \geq 2$. If $a = 1$, then $m_{a,b}^{(\lambda)}$ has simple poles at the $\lambda \in -\mathbb{N}$. The generalized function $m_{-}^{(\lambda)} := m_{2,0}^{(\lambda)} - m_{0,2}^{(\lambda)}$ has poles at $\lambda in -\mathbb{N}$.*

(b) $m_{0,0} := \mathrm{Res}_{\lambda=-\frac{5}{2}} m_{2,0}^{(\lambda)}$ is an $O(2,2)$-invariant Crofton distribution supported on $\Lambda_{0,0}(2)$.

(c) $m_{+} := m_{1,1}^{(-5/2)}$ is an $O(2,2)$-invariant Crofton distribution supported on $\overline{\Lambda_{1,1}(2)}$.

(d) $m_{-} := m_{-}^{(-5/2)}$ is an $O(2,2)$-invariant Crofton distribution supported on $\overline{\Lambda_{2,0}(2)} \cup \overline{\Lambda_{0,2}(2)}$.

Invariance follows from Lemma 11. Well-definedness involves working on the double cover $S^2 \times S^2$ of $\mathrm{Gr}_2(\mathbb{C}^2)$. They then computed the Klain functions of the valuations corresponding to $m_{0,0}$, m_{+}, and m_{-}, denoted $\phi_{0,0}$, ϕ_{+}, and ϕ_{-}, respectively. Ultimately, they found that

Theorem 36. *The Klain functions are given by:*

Valuation	*Klain function*
ϕ_-	$\frac{1}{3}(\kappa_2 - \kappa_0)$
ϕ_+	$-\frac{1}{3}\kappa_1$
$\phi_{0,0}$	$\frac{1}{3\pi}\kappa_1.$

The computations of the Klain functions spans the latter half of Section 8 in [9]. In particular, since by Theorem 2 in [9] we already know $\dim \mathrm{Val}_2^{-\infty}(\mathbb{R}^4)^{O(2,2)} = 2$, then we conclude any $O(2,2)$-invariant 2-homogeneous generalized even valuation has a Klain function in the span of $\{\kappa_1, \kappa_2 - \kappa_0\}$.

The relevence to $U(1,1)$ invariance is as follows: first, $\mathrm{Val}_2(\mathbb{R}^4)^{O(2,2)}$ inside $\mathrm{Val}_2(\mathbb{C}^2)^{U(1,1)}$ immediately implies $\dim \mathrm{Val}_2(\mathbb{C}^2)^{U(1,1)} \geq 2$. Furthermore, if we find some $\phi \in \mathrm{Val}_2(\mathbb{C}^2)^{U(1,1)}$, then to check ϕ is not $O(2,2)$-invariant (and hence that the dimension of $\mathrm{Val}_2(\mathbb{C}^2)^{U(1,1)}$ is larger than 2) it suffices to show its Klain function is not in $\mathrm{span}(\kappa_1, \kappa_2 - \kappa_0)$. Equivalently, if ϕ corresponds to the $U(1,1)$-invariant Crofton distribution m_ϕ, one must show the cosine transform $T_{2,2}m_\phi$ is not in $\mathrm{span}(\kappa_1, \kappa_2 - \kappa_0)$.

7.2 Proving $\dim \mathrm{Val}_2^{-\infty}(\mathbb{C}^2)^{U(1,1)} \geq 3$

We construct a $U(1,1)$-invariant generalized 2-homogeneous valuation that is not $O(2,2)$-invariant. First, we find a $U(1,1)$-invariant Crofton distribution, then show invariance by using its Cosine transform (i.e. the Klain function of the corresponding valuation). By Lemma 11, a $U(1,1)$-invariant Crofton distribution of degree 2 over $\mathrm{Gr}_2(\mathbb{C}^2)$ is given by a generalized function $f \in C^{-\infty}(\mathrm{Gr}_2(\mathbb{C}^2))$ that transforms by $M^*f = \psi_M^{-(n+1)/2} f$ for all $M \in U(1,1)$. In light of this identification, define the generalized function $m_L^{(\lambda)} \in C^{-\infty}(\mathrm{Gr}_2(\mathbb{C}^2))$, for $\lambda > 0$, by

$$\varphi \mapsto \int_{\mathscr{L}} \varphi(E) |\cos 2\tau(E)|^\lambda dE.$$

We claim $m_L^{(\lambda)}$ gives rise to a Crofton distribution.

Proposition 37. *(a) The generalized function $m^{(\lambda)}$ extends meromorphically to $\mathbb{C}$, and has simple poles at $\lambda \in -\mathbb{N}$.*

(b) The generalized function $m^{(-5/2)}$ corresponds to a $U(1,1)$-invariant Crofton distribution supported on the Lagrangian planes $\mathscr{L}$.

Proof. (a) For $\varphi \in C^\infty(\mathrm{Gr}_2(\mathbb{C}^2))$, let $\tilde{\varphi}$ denote its lift to the double cover $S^2 \times S^2$. Let dU denote the invariant probability measure on the maximal compact subgroup $K \leq U(1,1)$. Define

$$\Phi(z_2, w_2) := \int_k \tilde{\varphi}(U \cdot (z,w))dU, \quad z = (*, z_2, *),\ w = (*, w_2, *) \text{ such that } z, w \in S^2.$$

As K rotates the z and w spheres in $S^2 \times S^2$ independently of eachother around the z_2 and w_2 axes, Φ is well-defined.[5] Then we may express integration across $m_L^{(\lambda)}$ by, for some constant $c > 0$,

$$\langle m_L^{(\lambda)}, \varphi \rangle = c \int_{-1}^{1} (1 - z_2^2)^\lambda \Phi(z_2, 0) dz_2. \tag{21}$$

Note that in writing this equality, we used the facts that if $(z,w) \in \mathscr{L}$, then $w_2 = 0$, and also if (z,w) covers E, then $\cos 2\tau(E) = z_2^2 + w_2^2 - 1$ (see Section 6.3). Therefore, it suffices to show there is a meromorphic extension of $\tilde{m}_L^{(\lambda)}$, defined on $[-1,1]$ by $\Phi \mapsto \int_{-1}^{1}(1-x^2)^\lambda \Phi(x)dx$.

Note that near -1 (resp 1), the singularity of $(1-x^2)^\lambda$ behaves like $(1+x)^\lambda$ (resp. $(1-x)^\lambda$). Therefore, the meromorphic extension of $\tilde{m}_L^{(\lambda)}$ comes from that of the well-

[5] Put another way, we view $K = U(1) \times U(1)$, where the left (resp. right) copies of $U(1)$ are the stabilizers in $U(2)$ of z_2 (resp. w_2), and $dU = dU_1 dU_2$, where dU_i is the invariant probability measure on $U(1)$.

known generalized function x^λ. As x^λ is meromorphic on $\mathbb{C}$ with poles at $\lambda \in -\mathbb{N}$, the claim follows.

(b) We may now consider $m_L^{(-5/2)}$. Recall Lemma 11 to see that $U(p,q)$-invariance follows if we can show $M^* m_L^{(-5/2)} = \psi_M^{-5/2} m_L^{(-5/2)}$ for all $M \in U(p,q)$. In general, we compute

$$
\begin{aligned}
\langle M^* m_L^{(\lambda)}, \varphi\rangle &= \langle m_L^{(\lambda)}, |\psi_{M^{-1}}|^{5/2} \varphi \circ M^{-1}\rangle && \text{by definition of the pullback} \\
&= \int_{\mathscr{L}} |\psi_{M^{-1}}(E)|^{5/2} \varphi \circ M^{-1}(E) |\cos 2\tau(E)|^\lambda dE && \text{by definition of } m_L^{(\lambda)} \\
&= \int_{\mathscr{L}} |\cos 2\tau(ME)|^\lambda \varphi(E) dE && \text{changing variables} \\
&= \int_{\mathscr{L}} \psi_M^\lambda |\cos 2\tau(E)|^\lambda \varphi(E) dE && \text{by Proposition 10} \\
&= \langle \psi_M^\lambda m_L^{(\lambda)}, \varphi\rangle
\end{aligned}
$$

Note we also used that $\mathscr{L}$ is invariant under action of $U(1,1)$. So then $M^* m_L^{(\lambda)} = \psi_M^\lambda m_L^{(\lambda)}$, and we can apply Lemma 11 in the case $\lambda = -5/2$ to conclude $m_L^{(-5/2)}$ corresponds to a $U(1,1)$-invariant Crofton disribution.

□

So we have a $U(1,1)$-invariant Crofton distribution $m_L^{(-5/2)}$. Denote the corresponding valuation under the Crofton map by ϕ_L. We now prove

Proposition 38. *The Klain function of ϕ_L is linearly independant from κ_1 and $\kappa_2 - \kappa_0$.*

Note an immediate consequence is

Theorem 39. *The valuation ϕ_L is not $O(2,2)$-invariant, and so* $\dim \mathrm{Val}_2^{-\infty}(\mathbb{C}^2)^{U(1,1)} \geq 3$.

To prove Proposition 38, we will compute $\mathrm{Kl}_2(\phi_L)$ at various points. A priori, $\mathrm{Kl}_2(\phi_L)$ is a generalized section of Klain's bundle, and so it may not make sense to compute the Klain function at a given 2-plane E. However, such a computation is well-defined at any E

where we can show the Klain function is smooth. At such planes, we use Lemma 8 to say $\mathrm{Kl}_2(\phi_L)(E) = \langle m_L, |\cos(E,\cdot)|\rangle$. This motivates the necessity of our next proposition. Take $E_0 := V^0_{2,0} = \mathrm{span}(e_1, e_3) \in \Lambda_{2,0}(2)$. We claim

Proposition 40. *The generalized Klain function* $\mathrm{Kl}_2(\phi_L)$ *is smooth in some neighbourhood of* E_0.

In fact, it is possible to show $\mathrm{Kl}_2(\phi_L)$ is smooth at all $E \in \mathrm{Gr}_2(\mathbb{C}^2)$. The proof relies on fact that the wavefront sets of m_L and $|\cos(E,\cdot)|$ are disjoint (c.f. Proposition 3.16 in [9]). A complete explanation will be given in a later version of this project. For now, we assume the result so we may prove Proposition 38.

Proof of Proposition 38. For a contradiction, suppose $\mathrm{Kl}_2(\phi_L) = \alpha\kappa_1 + \beta(\kappa_2 - \kappa_0)$. By construction, the Klain function $\mathrm{Kl}_2(\phi_L)$ does not differentiate planes of signature $(2,0)$ from those of signature $(0,2)$. Therefore, we must have $\beta = 0$, and so $\mathrm{Kl}_2(\phi_L) = \alpha\kappa_1$. To get a contradiction, we show $\mathrm{Kl}_2(\phi_L)(E_0) \neq 0$, a goal which makes sense due to Proposition 40. By Lemma 8, we have $\mathrm{Kl}_2(\phi_L)(E_0) = \langle m_L, |\cos(E_0,\cdot)|\rangle$, so we first compute the cosine.

Given $F \in \mathrm{Gr}_2(\mathbb{C}^2)$, write $F = \mathrm{span}(u, v)$, where $\{u, v\}$ is an orthonormal basis with respect to the Euclidean structure P. The orthogonal projection of $\mathbb{C}^2$ onto F is then given by $\mathrm{Pr}_F = P(u,\cdot)u + P(v,\cdot)v$. In particular, the matrix of this projection as a map $E_0 \to F$ (with respect to the basis $\{e_1, e_3\}$ of E_0 and $\{u, v\}$ of F) is $\left(\begin{smallmatrix} u_1 & u_3 \\ v_1 & v_3 \end{smallmatrix}\right)$. Therefore, the volume of the orthogonal projection onto F of the unit square S_0 in E_0 is $u_1v_3 - u_3v_1$. In other words,

$$|\cos(E_0, F)| = \frac{\mathrm{vol}_2(\mathrm{Pr}_F(S_0))}{\mathrm{vol}_2(S_0)} = u_1v_3 - u_3v_1. \tag{22}$$

The plane E_0 lifts to $\left(\left(\begin{smallmatrix} 0 \\ 1 \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 \\ 1 \\ 0 \end{smallmatrix}\right)\right)$ in the double cover $S^2 \times S^2$. Then, supposing (z, w) covers F, equation (22) means the cosine function lifts to $S^2 \times S^2$ by

$$|\cos(E_0,(z,w))| = \frac{|z_2+w_2|}{2}.$$

Thus we have computed the cosine of the angle between E_0 and F. Now, using Proposition 40, Lemma 8, and equation 21 with the fact $\Phi(z_2,w_2) = \frac{|z_2+w_2|}{2}$, we find

$$\begin{aligned}
\mathrm{Kl}_2(\phi_L)(E_0) &= \langle m_L, |\cos(E_0,\cdot)|\rangle && \text{by Proposition 40 and Lemma 8}\\
&= \frac{c}{2}\int_{-1}^{1}(1-z_2^2)^{-(5/2)}|z_2|dz_1 && \text{by 21 and the definition of } m_L\\
&= \frac{c}{2}\langle \bar{m}_L^{(-5/2)}, |x|\rangle && \text{recalling } \bar{m}_L \text{ from the proof of Proposition 37}
\end{aligned}$$

Recall we defined $\bar{m}_L^{-(5/2)}$ by meromorphically extending $\bar{m}_L^{(\lambda)}$, where $\lambda > 0$. Therefore, to compute $\langle \bar{m}_L^{(-5/2)}, |x|\rangle$, it suffices to compute $m_L^{(\lambda)}$ at $|x|$ for $\lambda > 0$ large, and then evaluate the resulting expression in λ at $\lambda = -5/2$. We have, for $\lambda > 0$,

$$\begin{aligned}
\langle \bar{m}_L{}^{(\lambda)}, |x|\rangle &= \int_{-1}^{1}(1-x^2)^{\lambda}|x|dx && \text{by definition of } \bar{m}_L^{(\lambda)}\\
&= 2\int_0^1 (1-x^2)^{\lambda}xdx && \text{since the integrand is even}\\
&= \int_0^1 (1-u)^{\lambda}du && \text{changing variables } x^2 = u\\
&= \frac{1}{\lambda+1}.
\end{aligned}$$

Hence, setting $\lambda = -5/2$, we can conclude

$$\mathrm{Kl}_2(\phi_L)(E_0) = -\frac{2}{3} \neq 0,$$

as desired. This completes the proof. $\square$

8 Conclusion and Next Steps

Our goal for this project was to characterize $\mathrm{Val}^{-\infty}(\mathbb{C}^2)^{U(1,1)}$, the space of $U(1,1)$ and translation-invariant generalized convex valuations on $\mathbb{C}^2$. We achieved three results:

- The space $\mathrm{Val}^{-\infty}(\mathbb{C}^2)^{U(1,1)}$ is finite-dimensional (Theorem 26)
- Every 1-homogeneous $U(1,1)$-invariant generalized valuation is also $O(2,2)$-invariant (Theorem 31. Therefore, $\mathrm{Val}_1^{-\infty}(\mathbb{C}^2)^{U(1,1)}$ has been described completely in [9]
- There exists a 2-homogeneous $U(1,1)$-invariant generalized valuation that is not $O(2,2)$-invariant (Theorem 39). Therefore $\dim \mathrm{Val}_2^{-\infty}(\mathbb{C}^2)^{U(1,1)} \geq 3$.

Since the spaces of 0 and 4-homogeneous generalized valautions are spanned by the Euler characteristic and a Lebesgue measure, respectively, and the Alesker-Fourier transform gives an isomorphism $\mathrm{Val}_3^{-\infty}(\mathbb{C}^2)^{U(1,1)} \cong \mathrm{Val}_1^{-\infty}(\mathbb{C}^2)^{U(1,1)}$, all that remains is to find the dimension of the space of 2-homogeneous $U(1,1)$-invariant generalized valuations.

A first step in this goal is to carry out the following construction. Recalling the definition of m_L as a Crofton distribution given by integrating over the Lagrangian orbits $\mathscr{L}$, we define another family of distributions, $m_C^{(\lambda)}$, given by integrating over the complex orbits $\mathscr{C}$; that is, for $\lambda > 0$,

$$\langle m_C^{(\lambda)}, \varphi \rangle := \int_{\mathscr{C}} \varphi(E) |\cos 2\tau(E)|^{\lambda} dE, \quad \varphi \in C^{\infty}(\mathrm{Gr}_2(\mathbb{C}^2)).$$

In the double cover $S^2 \times S^2$ over $\mathrm{Gr}_2(\mathbb{C}^2)$, we found in Section 6.2 that $\mathscr{C}$ lifts to $(0, \pm 1, 0) \times S^2$. Using this fact, we can prove $m_C^{(\lambda)}$ extends meromorphically to $\lambda \in \mathbb{C}$. Then as in Proposition 37, it follows that $m_C := m_C^{-(5/2)}$ is a $U(1,1)$-invariant Crofton distribution. Once again, its wavefront set can be shown to be disjoint from that of $|\cos(E, \cdot)|$, and so we can then proceed as in Proposition 40 to conclude $\mathrm{Cr}_2(m_C)$ is a 2-homogeneous $U(1,1)$-invariant generalized valuation that is not $O(2,2)$-invariant. Showing $T_{2,2}m_C$ is in fact linearly independent from $\{\kappa_1, \kappa_2 - \kappa_0, T_{2,2}m_L\}$ will give $\dim \mathrm{Val}_2^{-\infty}(\mathbb{C}^2)^{U(1,1)} \geq 4$. The details of this construction will appear in a later version of this report.

A Representation Theory, the Gårding Topology, and Valuations

Here we give a definition and some relevant facts about the Gårding topology on $\mathrm{Val}^\infty(V)$. Our reference throughout will be [19], Wallach's *Real Reductive Groups I*. Let G be a finite-dimensional Lie group with finite number of connected components, and let (π, G, H) be a continuous representation of G in a Fréchet space H, in the sense that the map $G \times H \to H$ given by $(g, v) \mapsto \pi(g)v$ is continuous. We interchangeably write the action of G on H by $\pi(g)v = g \cdot v$. If, for $v \in H$, the map $G \to H$ given by $g \mapsto g \cdot v$ is smooth, we say v is a **smooth vector in** H. Denote the set of smooth vectors in H by H^∞. Gårding proved that

Theorem 41 (Gårding). *The set H^∞ is dense in H.*

Let $\mathfrak{g}$ be the Lie algebra of G. If $X \in \mathfrak{g}$, we set

$$\pi(X)v := \frac{d}{dt}\Big|_{t=0} \pi(\exp(tX))v.$$

In this way, $\pi(X)$ maps H^∞ to itself, and using Taylor's theorem we can show

$$\pi([X, Y]) = \pi(X)\pi(Y) - \pi(Y)\pi(X) \quad \text{for all } X, Y \in \mathfrak{g}.$$

This means $(\pi, \mathfrak{g}, H^\infty)$ is a representation of $\mathfrak{g}$. Now consider the **universal enveloping**

algebra $U(\mathfrak{g})$ of $\mathfrak{g}$, defined by taking the quotient of the tensor algebra $T(\mathfrak{g})$ over $\mathfrak{g}$ by the relation $X \otimes Y - Y \otimes X = [X, Y]$. A basis for $U(\mathfrak{g})$ is $X_1^{m_1} \cdots X_n^{m_n}$, where the X_i are a basis for $\mathfrak{g}$. In the context of Lie algebras, $U(\mathfrak{g})$ can be identified with the space of left-invariant differential operators on G. By the universal mapping property of $U(\mathfrak{g})$, we get that π extends to $U(\mathfrak{g})$. If $D \in U(\mathfrak{g})$, then set $\rho_{D,k}(v) := \|\pi(D)v\|_k$ for $v \in H^\infty$, where $\|\cdot\|_k$ are the semi-norms that define the topology on F. The $\rho_{D,k}$ are semi-norms, and we give H^∞ the topology induced by these semi-norms. This is the **Gårding topology**. We have

Theorem 42. *H^∞ is a Fréchet space, and (π, G, H^∞) is a smooth representation of G.*

A Fréchet representation (π, G, F) is called a **smooth representation** of G if $F = F^\infty$ as sets. Now let G be a real reductive Lie group. Then imbed G into $\mathrm{GL}_N(\mathbb{R})$ for some N as a closed subgroup invariant under transposition via an imbedding p. We define a norm $|\cdot|$ on G by

$$|g| := \max\{\|p(g)\|, \|p(g^{-1})\|\},$$

where $\|\cdot\|$ denotes the operator norm on $\mathbb{R}^N$. This allows us to make the following definition: we say a smooth representation has **moderate growth** if for each continuous semi-norm λ on F, there exists a continuous semi-norm ν_λ on F and real constant d_λ such that

$$\lambda(g \cdot v) \leq |g|^{d_\lambda} \nu_\lambda(v) \quad \text{for all } g \in G, v \in F.$$

The relevant result for representations with moderate growth is

Theorem 43. *If (π, G, H) is a continuous representation in a Banach space H, then (π, G, H^∞) has moderate growth.*

We say a Fréchet representation (π, G, H) of a real reductive Lie group G is **admissible** if its restriction to a maximal compact subgroup K of G contains an isomorphism class of any irreducible representation of K with at most finite multiplicity.

A.1 An Example of the Gårding Topology

Consider the case $G = (\mathbb{R}, +)$ and $H = C(\mathbb{R})$, with the toplogy generated by the norms $\|f\|_k := \sup\{|f(x)| : x \in [-k, k]\}$. Then H is a Fréchet space. The action of G on H is defined to be $x \cdot f := f(\cdot - x)$, which is continuous. It is not hard to see $H^\infty = C^\infty(\mathbb{R})$, the set smooth functions.

The Lie algebra of $\mathbb{R}$ is the one-dimensional space $\mathrm{span}_{\mathbb{R}}(\frac{\partial}{\partial x})$. The universal enveloping algebra is then spanned by the differential operators $\frac{\partial}{\partial x}$ for $n \in \mathbb{Z}_{\geq 0}$. Given $D = \frac{\partial^n}{\partial x^n}$, we have

$$\pi(D)f = \frac{d}{dt}\Big|_{t=0} \pi(\exp(tD))f = \frac{d}{dt}\Big|_{t=0} \left(1 + t\frac{\partial^n}{\partial x^n} + o(t^2)\right) f = f^{(n)},$$

where $f^{(n)}$ is the n-th derivative of f. So then, the semi-norms that generate the Gåding topology on $C^\infty(\mathbb{R})$ are $\rho_{D,k}(f) = \|f^{(n)}\|_k = \sup\{|f^{(n)}(x)| : x \in [-k, k]\}$. According to the theory described above, we get

- $C^\infty(\mathbb{R})$ is dense in $C(\mathbb{R})$, when equipped with the subspace topology.
- $C^\infty(\mathbb{R})$ is a Fréchet space with the topology given by the semi-norms $\rho_{D,k}$.
- $C^\infty(\mathbb{R})$ is a smooth representation of $\mathbb{R}$.

A.2 Intersection with Valuation Theory

Our discussion of the Gårding topology is necessary because is gives us the language and tools to prove the following lemma, which is used to prove Theorem 26.

Lemma 44. *Let V be a n-dimensional real vector space. For each $0 \leq k \leq n$, the space $\mathrm{Val}_k^{\pm}(V)$ is a smooth admissable Fréchet representation of $\mathrm{GL}(V)$ of moderate growth.*

Proof. By Alesker's irreducibility theorem, $\mathrm{Val}_k^{\pm}(V)$ is a continuous irreducible representation of $\mathrm{GL}(V)$ in a Banach space. Therefore, using Theorems 42 and 43, we find by its construction that $\mathrm{Val}_k^{\pm,\infty}(V)$, is a smooth Fréchet representation of moderate growth. According to Alesker, Remark 1.2.4 in [4], the representation $\mathrm{Val}(V)$ is admissible, and this implies the same holds for $\mathrm{Val}_k^{\pm,\infty}(V)$. Therefore the representation $\mathrm{Val}_k^{\pm,\infty}(V)$ satisfies the desired properties. □

B Some Definitions From Algebraic Geometry

Let G be an algebraic group. Recall the **derived series** of G is given recursively by

$$G^{(0)} := G, \quad G^{(n)} := [G^{(n-1)}, G^{(n-1)}] \text{ for } n \geq 1,$$

where $[G, G]$ is the commutator. We say G is **solvable** if its derived series terminates in the trivial group. A maximal, solvable, connected, algebraic subgroup B of an algebraic group G is called a **Borel subgroup** of G.

Now take G to be a connected algebraic reductive group over $\mathbb{R}$. We say a closed algebraic subgroup H of G is **spherical** if the action of H on G/B (where B is a Borel subgroup of G) has finitely many orbits. Equivalently, H is spherical if and only if the action of H on G/B has an open orbit.

The following lemma is used in an earlier section of this paper, and also serves as a useful example. While the result is standard, the proof is largely from [12].

Lemma 45. *Let $G = \mathrm{GL}_n(\mathbb{C})$. Then the subset B of upper-triangular matrices in G is a Borel subgroup.*

Proof. Note G is a Lie group, and B is a connected, algebraic Lie group. Then B is solvable if its Lie algebra $\mathfrak{b}$ is solvable, in the sense that the Lie commutator series

$$\mathfrak{b}^{(0)} := \mathfrak{b}, \quad \mathfrak{b}^{(n)} := [\mathfrak{b}^{(n-1)}, \mathfrak{b}^{n-1}] \text{ for } n \geq 1$$

eventually vanishes. Note $\mathfrak{b}$ is the set of all upper-triangular matrices (not necessarily invertible). If $A, B \in \mathfrak{b}$, then the k-th diagonal element of AB is $a_{kk}b_{kk}$, and the k-th diagonal element of BA is the same. Elements below the diagonal of AB and BA are 0. Therefore $[A, B]$, the general element of $\mathfrak{b}^{(1)}$, has the form

$$[A,B] = \begin{pmatrix} 0 & * & * & \cdots & * \\ 0 & 0 & * & \cdots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & * \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \tag{23}$$

Proceeding by induction, we see that $\mathfrak{b}^{(n)} = 0$, so $\mathfrak{b}$ and B are solvable. For maximality of B, suppose P between B and $\mathrm{GL}_n(\mathbb{C})$ is solvable. We can choose r and a_i for $i = 1, \ldots, r$ such P is the collection of all matrices A of the form

$$A = \begin{pmatrix} A_1 & * & * & \cdots & * \\ 0 & A_2 & * & \cdots & * \\ 0 & 0 & A_3 & \cdots & * \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & A_r \end{pmatrix},$$

where the A_i are arbitary elements of $\mathrm{GL}_{a_i}(\mathbb{C})$. Constuct the group homomorphsim

$$\varphi : P \to \mathrm{GL}_{a_1}(\mathbb{C}) \times \cdots \times \mathrm{GL}_{a_r}(\mathbb{C})$$
$$\varphi(A) := (A_1, \ldots, A_r).$$

The kernel $\ker(\varphi)$ is normal in $\mathrm{GL}_n(\mathbb{C})$, and since φ is surjective, then moreover

$$\mathrm{GL}_n(\mathbb{C})/\ker(\varphi) \cong \mathrm{GL}_{a_1}(\mathbb{C}) \times \cdots \times \mathrm{GL}_{a_r}(\mathbb{C}).$$

From general theory, P being solvable implies the quotient $\mathrm{GL}_n(\mathbb{C})/\ker(\varphi)$ is also solvable. So then the product $\mathrm{GL}_{a_1}(\mathbb{C}) \times \cdots \times \mathrm{GL}_{a_r}(\mathbb{C})$ is solvable, which means we necessarily have $a_i = 1$ for each i. Therefore $P \cong B$, and B is maximal. $\square$

References

[1] S. Alesker, *A Fourier type transform on translation invariant valuations on convex sets*, Israel J. Math. **181** (2011), 189-294.

[2] ______, *Description of translation invariant valuations on convex sets with solution of P. McMullen's conjecture*, Geom. Funct. Anal. **11** (2001), 244-272.

[3] ______, *The Multiplicative structure on polynomial continuous valuations*, Geom. Funct. Anal. **14** (2004), 1-26.

[4] ______, *Theory of valuations on manifolds, I. Linear spaces*, Israel J. Math **156** (2006), 311-339.

[5] S. Alesker and J. Bernstein, *Range characterization of the cosine transform on higher Grassmannians*, Adv. Math. **184** (004), 367-379.

[6] S. Alesker and D. Faifman, *Convex valuations invariant under the Lorentz group*, J. Differential Geom. **98** (2014), no. 2, 183-236.

[7] S. Alesker and J. H. G. Fu, *Integral Geometry and Valuations*, Advanced Courses in Mathematics - CRM Barcelona, Birkhäuser, Basel, 2014.

[8] A. Aizenbud, D. Gourevitch, and A. Minchenko, *Holonomicity of relative characters and applications to multiplicity bounds for spherical pairs*, Selecta Math. **22** (2016), no. 4, 2325-2345.

[9] A. Bernig and D. Faifman, *Valuation theory of indefinite orthogonal groups*, J. Funct. Anal. **273** (2017), 2167-2247.

[10] ______, *Generalized translation invariant valuations and the polytope algebra*, Adv. Math. **290** (2016), 36-72.

[11] D. Faifman, *Crofton formulas and indefinite signature*, Geom. Funct. Anal. **27** (2017), 489-540.

[12] M. Garcia, *General parabolic subgroups of* $\mathrm{GL}_n(\mathbb{C})$, Massachusetts Institute of Technology, May 16, 2005. Lecture handout.

[13] H. Hadwiger, *Vorlesungen über Inhalt, Oberfläche und Isoperimetrie*, Springer-Verlag, Berlin-Göttingen-Heidelberg, 1957 (German).

[14] D. A. Klain, *Even valuations on convex bodies*, Trans. Amer. Math. Soc. **352** (2000), no. 1, 71-93.

[15] J. Lee, *Introduction to Smooth Manifolds*, 2nd ed., Springer-Verlag, New York, 2012.

[16] P. McMullen, *Valuations and Euler-type relations on certain classes of convex polytopes*, Proc. London Math. Soc. **35** (1977), no. 1, 113-135.

[17] B. Simon, *Orthogonal Polynomials on the Unit Circle*: *Part 2: Spectral Theory*, Vol. 54, Colloquium Publications, Pasadena, 2005.

[18] H. Tasaki, *Generalization of Kähler angle and integral geometry in complex projective spaces*, Steps in Differential Geometry (Debrecen, 2000), Inst. Math. Inform., Debrecen, 2001, pp. 349-361.

[19] N. R. Wallach, *Real Reductive Groups I*, Academic Press, San Diego, 1988.