Basic forms on foliated manifolds

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Main Result

Given a foliated manifold $(M, \mathcal{F})$, consider:

- the complex of basic differential forms $\Omega^\bullet_{\text{basic}}(M)$, consisting of $\alpha \in \Omega^\bullet(M)$ such that $\iota_X \alpha = 0$ and $\mathcal{L}_X \alpha = 0$ for all vector fields X tangent to $\mathcal{F}$
- the complex of diffeological differential forms $\Omega^\bullet(M/\mathcal{F})$, consisting of diffeological differential forms on the leaf space carrying the quotient diffeology.

Theorem. The pullback by the quotient map $\pi : M \to M/\mathcal{F}$ gives an isomorphism $\Omega^\bullet(M/\mathcal{F}) \to \Omega^\bullet_{\text{basic}}(M)$.

Diffeology

Definitions

- A **diffeology** on a set X is a set $\mathcal{D}$ of maps from open subsets U of Euclidean spaces to X, with members called *plots*, such that
  - Constant maps are plots.
  - If $p : U \to X$ is such that about each point in U there is open $V \subseteq U$ such that $p|_V$ is a plot, then p is a plot.
  - If $p : U \to X$ is a plot and V is an open subset of Euclidean space, then F^*p is a plot for all smooth $F : V \to U$.

  We call $(X, \mathcal{D})$ a **diffeological space**.
- A **diffeological differential k-form** α on $(X, \mathcal{D})$ is an assignment to each plot $p : U \to X$ a k-form $\alpha(p) \in \Omega^k(U)$, such that for every open subset V of Euclidean space and smooth map $F : V \to U$,

  $$\alpha(p \circ F) = F^*(\alpha(p)).$$

  Diffeological differential forms assemble into a de Rham complex $\Omega^\bullet(X)$.

Examples

- Smooth maps $U \to M$ form a diffeology on a manifold M.
- Diffeological forms on M are identified with usual differential forms.
- The quotient space $X/\sim$ for an equivalence relation $\sim$ carries the finest diffeology in which the quotient map $\pi : X \to X/\sim$ is diffeologically smooth (i.e. pullbacks of π by plots of X are plots).

Key Result

Proposition 1. The quotient map $\pi : X \to X/\sim$ induces an injection $\pi^* : \Omega^\bullet(X/\sim) \to \Omega^\bullet(X)$. Its image consists of forms $\alpha \in \Omega^\bullet(X)$ such that for plots $p_1, p_2 : U \to X$ with $\pi \circ p_1 = \pi \circ p_2$,

$$p_1^*\alpha = p_2^*\alpha.$$

See [1].

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Groupoids

Definitions

- A **groupoid** G is a small category $G_1 \rightrightarrows G_0$ with invertible morphisms. Denote its source s and target t. It is **Lie** if G_0 and G_1 are manifolds (except G_1 need not be second-countable or Hausdorff), the structure maps are smooth, and s is a submersion.
- $\alpha \in \Omega^\bullet(G_0)$ is **G-basic** if $s^*\alpha = t^*\alpha$. These forms assemble into a de Rham complex $\Omega^\bullet_{G_1 \rightrightarrows G_0}(G_0)$.
- A functor $f : G \to H$ of Lie groupoids is a **refinement** if
  - the following is a fibered square

    $$\begin{array}{ccc} G_1 & \xrightarrow{f_1} & H_1 \\ \downarrow{\scriptstyle (s,t)} & & \downarrow{\scriptstyle (s,t)} \\ G_0 \times G_0 & \xrightarrow{(f_0, f_0)} & H_0 \times H_0 \end{array}$$

  - the map $s \circ pr_2 : G_0 \ {}_f\!\times_t H_1 \to H_0$ is a surjective submersion.
- G and H are **Morita equivalent** if there are refinements from a Lie groupoid K to G and to H.

Examples

- A foliated manifold $(M, \mathcal{F})$ admits the (Lie) **holonomy groupoid** $\mathrm{Hol}(M, \mathcal{F})$, with objects the points of M, arrows the *holonomy classes* of leaf-paths, and orbits the leaves.
- Given $\varphi : N \to G_0$, we have the pullback Lie groupoid $\varphi^*(G)$ over N when $t \circ \mathrm{pr}_1 : G_1 \ {}_s\!\times_\varphi N \to G_0$ is a submersion.
  - The pullback of $\mathrm{Hol}(M, \mathcal{F})$ by a complete transversal $\iota : S \hookrightarrow M$ to $\mathcal{F}$, denoted $\mathrm{Hol}_S(M, \mathcal{F})$, is an *étale* (i.e. $\dim G_1 = \dim G_0$) and *effective* Lie groupoid Morita equivalent to $\mathrm{Hol}(M, \mathcal{F})$.

Key Results

Proposition 2. See [4]. A refinement $f : G \to H$ induces

- An isomorphism $f_0^* : \Omega^\bullet_{H_1 \rightrightarrows H_0}(H_0) \to \Omega^\bullet_{G_1 \rightrightarrows G_0}(G_0)$.
- A diffeological diffeomorphism $[f] : G_0/G_1 \to H_0/H_1$.

Proposition 3. $\mathrm{Hol}(M, \mathcal{F})$-basic forms are exactly $\mathcal{F}$-basic forms.

Pseudogroups

Definitions

- A **pseudogroup** on M is a set P of diffeomorphisms of open subsets of M such that:
  - $\mathrm{id}|_U \in P$ for every open U.
  - If $f, f' \in P$, then so are $f' \circ f$ and f^{-1}.
  - If $f : U \to U'$ is a diffeomorphism, $\{U_i\}$ covers U, and every $f|_{U_i} \in P$, then $f \in P$.
- For étale $G_1 \rightrightarrows G_0$, the set of *bi-submersions* $\mathbf{\Psi(G)}$ is a pseudogroup on G_0.
- For a pseudogroup P, we get a Lie groupoid $\mathbf{\Gamma(P)}$ over M with arrows the germs of elements of P.

Key Results

Proposition 4. The pseudogroup $\Psi(\mathrm{Hol}_S(M, \mathcal{F}))$ is countably generated. See [3].

Proposition 5. For étale effective G, basic forms are exactly $\Psi(G)$-invariant forms.

The Proof

Step 1: Notation and Reformulation

Fix the following:

- A foliated manifold $(M, \mathcal{F})$, with complete transversal $\iota : S \hookrightarrow M$.
- $H := \mathrm{Hol}(M, \mathcal{F})$, and $G := \mathrm{Hol}_S(M, \mathcal{F})$.
- $P := \Psi(\mathrm{Hol}_S(M, \mathcal{F}))$.

By **Proposition 3**, and the fact the orbits of H are the leaves, we may instead prove π^* gives an isomorphism $\Omega^\bullet(M/\mathcal{F}) \to \Omega^\bullet_{H_1 \rightrightarrows M}(M)$.

Step 2: Well-posedness and injectivity

- We use the second clause of **Proposition 1** to show that the pullback $\pi^* : \Omega^\bullet(M/\mathcal{F}) \to \Omega^\bullet(M)$ has image contained in $\Omega^\bullet_{H_1 \rightrightarrows M}(M)$.
- We use the first clause of **Proposition 1** to see that π^* is an injection.

Therefore it only remains to prove π^* surjects onto $\Omega^\bullet_{H_1 \rightrightarrows M}(M)$.

Step 3: Surjectivity

The complete transversal $\iota : S \hookrightarrow M$ induces a refinement $\iota : G \to H$. **Proposition 2** gives a commutative diagram:

$$\begin{array}{ccc} \Omega^\bullet_{H_1 \rightrightarrows M}(M) & \xrightarrow{\iota^* \ \cong} & \Omega^\bullet_{G_1 \rightrightarrows S}(S) \\ {\scriptstyle \pi^*}\uparrow & & \uparrow{\scriptstyle \pi_G^*} \\ \Omega^\bullet(M/\mathcal{F}) & \xrightarrow{[\iota]^* \ \cong} & \Omega^\bullet(S/G_1) \end{array} \qquad (1)$$

We prove π_G^* is onto. Take G-basic α on S. By **Proposition 5**, α is P-invariant. By **Proposition 1**, we must show for all smooth plots $p_1, p_2 : U \to S$ with $\pi_G \circ p_1 = \pi_G \circ p_2$, we have $p_1^*\alpha = p_2^*\alpha$.
Because P is countably generated by **Proposition 4**, we can choose $\{f_I\} \subseteq P$ countable with:

(i) $U = \bigcup_I C_I$, where $C_I := \{u \in U \mid p_1(u) = f_I \circ p_2(u)\}$.

(ii) $p_1 = f_I \circ p_2$ on $\mathrm{int}\, C_I$.

The first point (i), together with the fact the C_I are closed in U, implies through the Baire category theorem that $U = \overline{\bigcup_I \mathrm{int}\, C_I}$. The second point (ii), combined with P-invariance of α, gives that $p_1^*\alpha = p_2^*\alpha$ on $\mathrm{int}\, C_I$. By continuity, $p_1^*\alpha = p_2^*\alpha$ on the closure $\overline{\mathrm{int}\, C_I}$. Hence equality holds on U. This completes the proof of surjectivity, hence the theorem.

References

[1] P. Iglesias-Zemmour, *Diffeology*, Mathematical Surveys and Monographs, vol. 185, American Mathematical Society, Providence, 2013.

[2] Y. Karshon and J. Watts, *Basic forms and orbit spaces: a diffeological approach*, SIGMA Symmetry Integrability Geom. Methods Appl. **12**, 19 pp.

[3] I. Moerdijk and J. Mrcŭn, *Introduction to foliations and Lie groupoids*, Cambridge Studies in Advanced Mathematics, vol. 91, Camebridge University Press, Cambridge, 2003.

[4] J. Watts, *The orbit space and basic forms of a proper Lie groupoid*, 2015. preprint at `arXiv:1309.3001v3`.