In these exercises, we will work to understand constructions from Euclid.

Exercise 1. Below is a diagram, adapted from Euclid IV. 5.:

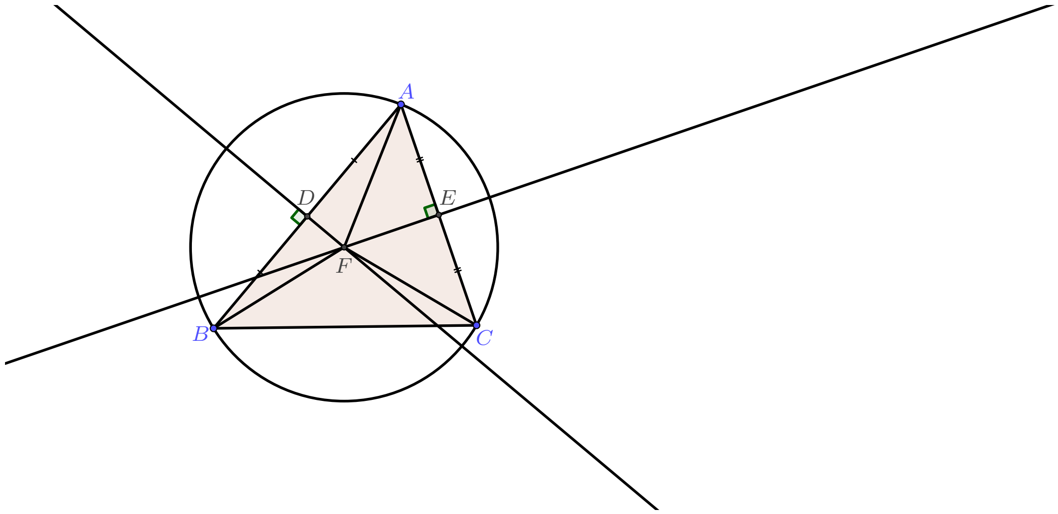

Figure 1: A diagram from Euclid IV. 5.

(a) Assume we began with the points A, B, and C. Describe the construction which produces this diagram.

(b) For the construction you outlined above, are there possible configurations besides that in Figure 1?

(c) What does this construction accomplish?

Exercise 2. Consider the following construction given in Euclid I. 22:

> Let the three given straight lines be A, B, C, and of these let two taken together in any manner be greater than the remaining one...
> Let there be set out a straight line DE, terminated at D but of infinite length in the direction of E, and let DF be made equal to A, FG be equal to B, and GH equal to C [I. 3].
> With centre F and distance FD let the circle DKL be described; again, with centre G and distance GH let the circle KLH be described; and let KF, KG be joined...

(a) Draw a diagram illustrating this construction.

(b) What does this construction accomplish?

In this exercise, we will work through the construction in Euclid III. 25:

Given a segment of a circle, to describe the complete circle of which it is a segment.

Exercise 3.

(a) We will let ourselves use Euclid III. 9:

If a point be taken within a circle, and more than two equal straight lines fall from the point on the circle, the point taken is the centre of the circle.

How can we use this Proposition to confirm some point is a centre of the circle?

(b) Suppose the segment is as pictured. Complete the partial construction to find the centre. (Hint: make isosceles triangles using Euclid I. 23.).

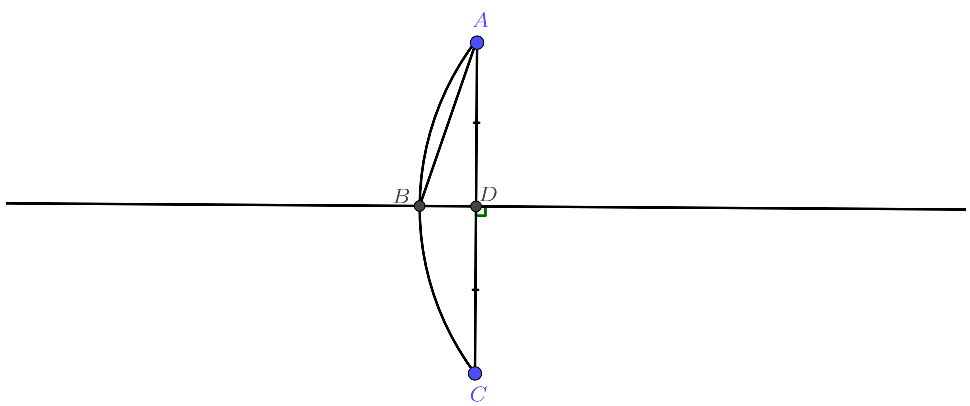

Figure 2: A partial construction for Euclid III. 25.

(c) Why does your construction work?

(d) There are two other cases, depending on the angles ABD and BAD. What are these cases, and why are there not more? (Hint: what kind of relation is "greater than"?).

(e) Why is the following argument incomplete:

- Take three distinct points on the segment, A, B, and C.
- Construct the circle circumscribing the triangle ABC, using Euclid IV. 5.
- This circle is the desired one.

This is inspired by Exercise 6.3 in Hartshorne. A *projective plane* is a set of points and lines (each line is a collection of points) satisfying these axioms:

P1 Any two distinct points lie on a unique line.

P2 Any two lines meet in at least one point.

P3 Every line contains at least three points.

P4 There exist three non-collinear points.

Remember: for a model of these axioms, a "point" can be any type of object, as long as we define "line" in a way that points and lines satisfy the axioms.

Exercise 4.

(a) For each proposed model below, state the axiom(s) it does not satisfy, and explain why.

(i) A point is any point in the plane. A line is any line in the plane.

(ii) A point is any point in the plane. A line is any line in the plane through the origin.

(iii) A point is any student in this room. A line is the collection of all students in this room.

(iv) A point is a vertex $A, \ldots, F$, as pictured below. A line is the set of all vertices within a given straight line segment, or the set of all vertices within the circle.

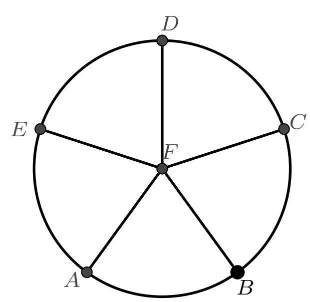

Figure 3: A failed model of a projective plane.

Thus none of the proposed models constitute a projective plane.

(b) Draw a model of a projective plane on seven vertices.

In this tutorial, we will examine a projective completion of a finite model of incidence geometry.

Exercise 5. Let P be the four point plane shown below.

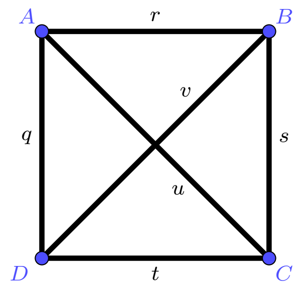

Figure 4: A four point plane

(a) In class, we saw that P satisfies the axioms of incidence geometry, and Euclid's parallel postulate. Let $\hat{P}$ be its projective completion. List the points of $\hat{P}$, and the lines of $\hat{P}$. (Optional: try to draw $\hat{P}$).

(b) Verify directly that $\hat{P}$ satisfies the axioms of incidence geometry, and moreover that every line contains at least three points, and that any two lines meet. (Hint: you already know that P is a model of incidence geometry). This proves $\hat{P}$ is a projective plane.

(c) Add lines through the points below so that $\hat{P}$ is isomorphic to the model you create. Give the isomorphism.

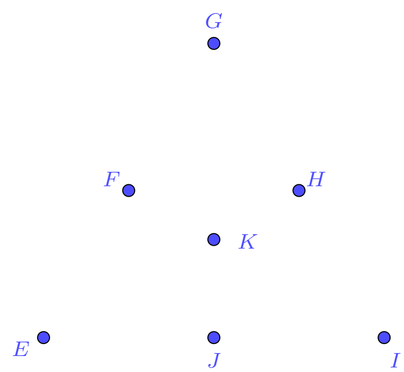

Figure 5: The points of a model you will complete

Exercise 6. Let: W be the plane $z = 0$; P be the plane $y = 3$; ℓ be the line parametrized by $x = 0.5$, $y = t$, $z = 0$; $O := (0, 0, 2)$.

(a) Given a point $A = (0.5, t, 0)$ in ℓ, find the coordinates of the intersection of the line $\overleftrightarrow{O\tilde{A}}$ and the plane P. Call this new point $\tilde{A}$.

(b) As $t \to \infty$, what point (if any) does $\tilde{A}$ approach in P? What about as $t \to -\infty$?

(c) Repeat parts (a) and (b), only replace ℓ with a line m in W of your choosing, such that

(i) m is parallel to ℓ;

(ii) m is parallel to the line $x = t, y = t, z = 0$;

(iii) m is parallel to the line $x = t, y = 0, z = 0$.

(d) Fill in the blanks: The points in P that are not of the form $\tilde{A}$ for any A in W lie in the intersection of P with the plane ______. This is called the *horizon line* in P. The lines in W whose counterparts (when they exist) in P do not approach the horizon line are all parallel to ______.

Exercise 7. Let P be the plane $x = 1$, and W be the plane $x + z = 1$. Consider the central projection (from the origin) of W onto P, denoted

$$p = (x, y, z) \mapsto \tilde{p} = (\tilde{x}, \tilde{y}, \tilde{z}).$$

(a) Write P and W in vector form. Use this to get an expression for $\tilde{p}$.

(b) Let $p = (x, y, z) = \left(\frac{\cos(\theta)+1}{2}, \frac{\sin(\theta)}{\sqrt{2}}, \frac{1-\cos(\theta)}{2}\right)$. Verify that $p \in W$, and that $x^2 + y^2 + z^2 = 1$. This shows that p traces out a circle.

(c) For p as in (b), write down $\tilde{p}$ using (a). Verify that $\tilde{z} = \frac{1}{2}\tilde{y}^2$. This shows that $\tilde{p}$ traces out a parabola.

(d) Using the language of perspective geometry, explain why a "closed" circle in W corresponds to an "open" parabola in P (Hint: where is the horizon in W?)

Exercise 8. We consider the *Klein disk* model of incidence geometry.

- Its points are the points inside an open unit disk Δ (situated inside the plane).
- Its lines are the chords inside Δ.

This satisfies the axioms I1–I3 of incidence geometry. It also satisfies the betweenness axioms B1–B4, where "B is between A and C" means the same thing it does in the usual model of Euclidean geometry.

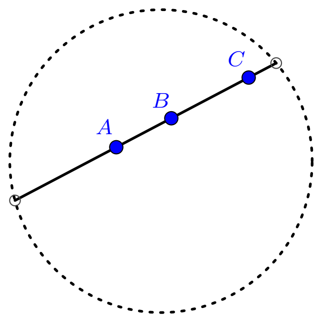

Figure 6: The Klein disk, with a line through A and B (and C). Here B is between A and C. Remember the boundary of the disk is not included, so the line has no "endpoints."

(a) Which parallel property does this model satisfy: Euclidean, Hyperbolic, or Elliptic?

(b) True or false (and justify): given three parallel lines, there is always a line passing through all three of them. (Hint: axiom B4 is relevant).

(c) Suppose we declared segment $\overline{AB}$ to be congruent to segment $\overline{CD}$ if they have the same Euclidean length (i.e. they are congruent in the usual model of Euclidean geometry). Which congruence axioms C1–C3 does this proposed relation satisfy?

Exercise 9. Assume the incidence axioms, the betweenness axioms, and the congruence axioms.

(a) Let A, B, and C be distinct points, and let U denote the interior of the angle $\angle ABC$. Let D and D' be distinct points inside U. Prove that if E is between D and D', then E is in U. (This is how you prove U is *convex*).

(b) Let O and A be distinct points. Fill in the blank and prove the statement:

$$\text{There are exactly } \underline{\hspace{2em}} \text{ points } B \text{ on the line } \overleftrightarrow{OA} \text{ satisfying: } \overline{OB} \cong \overline{OA}.$$

Exercise 10. Given two distinct points $p, q \in \mathbb{R}^2$, the points on the segment $\overline{pq}$ are precisely the points of the form

$$tp + (1-t)q, \quad \text{where } 0 < t < 1.$$

- Prove $\frac{p+q}{2}$ is between p and q.
- Prove $\overline{(1,2)(2,1)}$ does not meet $\overline{(2,0)(3,1)}$. (Hint: does it meet the line $x - y = 2$?)

Consider $\mathbb{Q}^2 = \{(x, y) \in \mathbb{R}^2 \mid x, y \in \mathbb{Q}\}$, with the interpretations of "point", "line", "incidence", "betweenness", and "congruence" adapted from their meaning in $\mathbb{R}^2$. The incidence and betweenness axioms hold.

Exercise 11. For each property below, either prove $\mathbb{Q}^2$ satisfies it, or use an example to show the property fails.

(a) The circle-circle intersection property (remember HW2).

(b) The first congruence axiom (segment transportation along any ray).

(c) The Dedekind property.

(d) The Archimedean property.

Exercise 12. Does the model $\mathbb{Q}^2$ show that the circle-circle intersection property is independent of the axioms of neutral geometry?

Exercise 13. Rewrite each sentence without the indicated word:

(a) The angle $\angle ABC$ is a right angle.

(b) The line $\overleftrightarrow{AB}$ is perpendicular to $\overleftrightarrow{CD}$.

(c) The ray $\overrightarrow{BD}$ bisects the angle $\angle ABC$.

(d) The point C is the midpoint of the segment $\overline{AB}$.

Exercise 14. Below is a "proof" of Euclid I. 20, which states: in a triangle, any two sides taken together are greater than the third. Find the errors and fill in any gaps you feel are present. (Hint: there are at least four things to change).

Proof. Let ΔABC be the triangle. Take the point D such that $\overline{AD} \cong \overline{AB}$ (using C-1). Then $\angle ADC \cong \angle ACD$. Because $\overline{AC}$ is inside angle $\angle BCD$, we have $\angle ADC \cong \angle ACD < \angle BCD$. Therefore, in triangle ΔDBC, the angle opposite $\overline{BD}$ is larger than the angle opposite $\overline{BC}$, hence $\overline{BD} > \overline{BC}$. But $\overline{BD} \cong \overline{BA} + \overline{AD}$, which proves the statement. □