In each exercise, we consider affine lines in $\mathbb{R}^2$ in a different way.

Bonus note: Exercise 1 generalizes to describe affine hyperplanes in $\mathbb{R}^n$. Exercise 2 generalizes to describe lines in $\mathbb{R}^n$.

Exercise 1. An affine line in $\mathbb{R}^2$ is given by (the solution set of) an equation of the form

$$ax + by = c, \quad \text{for some } a, b, c \in \mathbb{R}, \text{ with } (a, b) \neq (0, 0). \tag{1}$$

Denote the set of affine lines in $\mathbb{R}^2$ by $\mathrm{AGr}_1(\mathbb{R}^2)$.

(a) What condition (equation) can we impose on a, b without discarding any lines from consideration? (Hint: $a^2 + b^2 \neq 0$).

(b) If

$$ax + by = c \text{ and } a'x + b'y = c'$$

describe the same line, where a, b and a', b' satisfy your condition in (a), can you relate (a, b) to (a', b'), and c to c'?

(c) Describe a function $f : S^1 \times \mathbb{R} \to \mathrm{AGr}_1(\mathbb{R}^2)$. Is f surjective? For a line $\ell \in \mathrm{AGr}_1(\mathbb{R}^2)$, how large is $f^{-1}(\ell)$?

Exercise 2. An affine line in $\mathbb{R}^2$ is given by choosing a line L through the origin, choosing a point $(x_0, y_0) \in \mathbb{R}^2$, and translating L to go through (x_0, y_0).

(a) If L and (x_0, y_0) and L' and (x_0', y_0') describe the same line, can you relate L and L', and (x_0, y_0) and (x_0', y_0')?

(b) The set of lines through the origin is $\mathbb{R}P^1$. Describe a function $g : \mathbb{R}P^1 \times \mathbb{R}^2 \to \mathrm{AGr}_1(\mathbb{R}^2)$. For a line $\ell \in \mathrm{AGr}_1(\mathbb{R}^2)$, describe $g^{-1}(\ell)$.

Exercise 3. A non-horizontal line in $\mathbb{R}^2$ is given by (the solution set of) an equation of the form (1), except also $a \neq 0$.

(a) Exercise 1 (a), except only impose a condition on a (Hint: $a \neq 0$).

(b) Exercise 1 (b).

(c) Describe a function $h : \mathbb{R} \times \mathbb{R} \to \mathrm{AGr}_1(\mathbb{R}^2)$. Is h surjective? For a line $\ell \in \mathrm{AGr}_1(\mathbb{R}^2)$, how large is $h^{-1}(\ell)$?

(d) Could h be a coordinate chart for $\mathrm{AGr}_1(\mathbb{R}^2)$?

Exercise 4. Let M be an n-manifold, with maximal atlas $\mathcal{A} = \{(U, \varphi)\}$, and let X be a set. Suppose $f : M \to X$ is a bijection. Consider the collection given by

$$\mathcal{B} := \{(f(U), \varphi \circ f^{-1})\},$$

where the f^{-1} above is understood to be the restriction $f^{-1} : f(U) \to U$.

(a) Prove the elements of $\mathcal{B}$ are charts of X.

(b) Prove that X, equipped with the maximal atlas generated by $\mathcal{B}$, is a manifold. (Hint: the countability and Hausdorff conditions are easier).

(c) Let $M = \mathbb{R}$, with the standard maximal atlas, and $X = \mathbb{R}$. Can you give an example of a bijection $f : \mathbb{R} \to \mathbb{R}$ such that $\mathcal{B} \neq \mathcal{A}$? (Hint: f cannot be a diffeomorphism $(\mathbb{R}, \mathcal{A}) \to (\mathbb{R}, \mathcal{A})$).

(d) Bonus: does $\mathbb{R}^2$ have the structure of a 1-dimensional manifold?

Exercise 5. Let X, Y, and Z be sets, and $f : X \to Y$ be a function. We denote by $\mathrm{Fun}(Z, X)$ the set of functions from Z to X. Consider the function (the *pushforward* by f) defined by

$$f_* : \mathrm{Fun}(Z, X) \to \mathrm{Fun}(Z, Y), \quad f_*(g) := f \circ g.$$

(a) Does it make sense to ask if, for all $\lambda \in \mathbb{R}$,

$$f_*(\lambda g_1 + g_2) = \lambda f_*(g_1) + f_*(g_2)?$$

If so, prove it is true, or give a counter-example. If not, how could we re-write the set-up for the statement to make sense, and for it to be true?

(b) Same question as (a), except for the statement

$$(\lambda f_1 + f_2)_* = \lambda (f_1)_* + (f_2)_*.$$

Exercise 6. For this exercise, we assume some familiarity with complex numbers. The unit sphere in $\mathbb{C}^2$ is $S^3 := \{(z,w) \mid |z|^2 + |w|^2 = 1\}$. This is a submanifold. We are going to decompose it into a union of circles.

(a) Recall that complex projective space, $\mathbb{C}P^1$, is the quotient of $\mathbb{C}^2 \smallsetminus \{\mathbf{0}\}$ by the relation: $(z_1, w_1) \sim (z_2, w_2)$ if and only if $(z_1, w_1) = \lambda(z_2, w_2)$ for some $\lambda \in \mathbb{C}$. The equivalence class of (z, w) is denoted $[z : w]$. Just like $\mathbb{R}P^n$, $\mathbb{C}P^1$ is a manifold. Consider the map

$$\pi : S^3 \to \mathbb{C}P^1, \quad \pi(z,w) := [z : w].$$

In one sentence, why is this a smooth map? Is it surjective?

(b) For $\ell \in \mathbb{C}P^1$, describe $\pi^{-1}(\ell)$. (Hint: start by fixing (z_0, w_0) with $\pi(z_0, w_0) = \ell$).

(c) Describe a map $S^1 \to \pi^{-1}(\ell)$. Is your map invertible? (Hint: $S^1 \subseteq \mathbb{C}$).

The decomposition $S^3 = \bigsqcup_{\ell \in \mathbb{C}P^1} \pi^{-1}(\ell)$ is called the *Hopf fibration*. There is more information in the textbook.

Exercise 7. In this exercise, we look at smooth maps into submanifolds.

(a) Let $\varphi = (\varphi^1, \ldots, \varphi^m) : \mathbb{R}^m \to \mathbb{R}^m$ be a diffeomorphism. Set $S := \{\varphi^{k+1} = \cdots = \varphi^m = 0\}$. Is this a submanifold of $\mathbb{R}^m$? If so, what are some charts?

(b) Suppose $f : N \to \mathbb{R}^m$ is a smooth map from an n-manifold N, and $f(N) \subseteq S$. Prove the map

$$f|^S : N \to S, \quad f|^S(q) = f(q)$$

is smooth. (Bonus: do the same thing, except now assume that S is an arbitrary submanifold of a manifold M.)

(c) (Bonus) The sets $X_- := \{(x, 0) \mid x < 0\}$, $X_+ := \{(x,0) \mid x > 0\}$, and $Y := \{(0, y)\}$ are all manifolds. Therefore their disjoint union $Q := X_- \sqcup X_+ \sqcup Y$ has a manifold structure. Prove, however, that this manifold structure cannot be a submanifold structure induced from $\mathbb{R}^2$. (Hint: you should use (b)).

We are going to practice finding the rank of a smooth map through investigating some properties of an interesting immersion, called the *irrational winding of the torus*. Before we begin, you might find the following fact useful: if A and B are linear maps (that can be composed), then

$$\operatorname{rank} AB \leq \operatorname{rank} B \quad \text{and also} \quad \operatorname{rank} AB \leq \operatorname{rank} A.$$

Now fix an irrational number α, and let

$$f : \mathbb{R} \to S^1 \times S^1, \quad f(t) := (\cos(t), \sin(t), \cos(\alpha t), \sin(\alpha t)).$$

Exercise 8.

(a) Show that f has maximal rank at $t = 0$, either by viewing f as a map into $\mathbb{R}^4$, or by recalling that $(x, y) \mapsto \frac{x}{1-y}$ is a chart of S^1. In fact, f is an immersion.

(b) Prove that f is injective (Hint: the first two coordinates of $f(t)$ are the point on the circle with angle t radians to the x-axis).

(c) Did we prove a theorem that says that $f(\mathbb{R})$ is a submanifold of $S^1 \times S^1$?

Let $\Lambda = f(\mathbb{R})$. Because f is injective, we can give Λ the atlas generated by the chart

$$(f|^{\Lambda})^{-1} : \Lambda \to \mathbb{R}.$$

Let Λ_f denote this manifold.

Exercise 9.

(a) Is the inclusion $\iota_f : \Lambda_f \to S^1 \times S^1$ an immersion?

(b) The following property holds:

$$\text{if } g : N \to S^1 \times S^1 \text{ is a smooth map, and } g(N) \subseteq \Lambda, \text{ then } g|^{\Lambda_f} : N \to \Lambda_f \text{ is smooth.} \quad (*)$$

Using this property, prove that: if $\Lambda_?$ is some manifold whose underlying set is Λ, and also the inclusion $\iota_? : \Lambda_? \to S^1 \times S^1$ is an immersion, then the identity map $\mathrm{id} : \Lambda_? \to \Lambda_f$ is a diffeomorphism.

Bonus information: the image Λ is a dense subset of $S^1 \times S^1$; it is not a submanifold. Nevertheless, we saw in Tutorial 3 that property (*) for Λ_f is exactly the same property that all submanifolds have. This suggests that it still makes sense to think of Λ_f as "submanifold-like." Indeed, we call subsets like Λ *weakly-embedded submanifolds*.

Every tangent vector $v \in T_xM$ is a linear function $C^\infty(M) \to \mathbb{R}$ that is, by definition, represented by a curve $\gamma : \mathbb{R} \to M$, with $\gamma(0) = x$, such that

$$v(f) = \frac{d}{dt}\Big|_{t=0} f(\gamma(t)).$$

Let's write $v = [\gamma]$ for "v is represented by γ." If $F : M \to N$ is a smooth function, then its tangent map is the linear map $T_xF : T_xM \to T_{F(x)}N$, which is the linear function $C^\infty(N) \to \mathbb{R}$ given by

$$(T_xF(v))(g) = v(g \circ F).$$

If $v = [\gamma]$, then $(T_xF(v))(g) = \frac{d}{dt}\Big|_{t=0} g(F(\gamma(t)))$. So then $T_xF(v) = [F \circ \gamma]$.

Exercise 10. Let $F : \mathrm{Mat}(2;\mathbb{R}) \to \mathrm{Mat}(2;\mathbb{R})$ be defined by

$$F(A) := A\Phi A^\top,$$

where $\Phi = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$.

(a) If V is a vector space, and $x \in V$, how can you associate to $v \in V$ a vector in T_xV? Is this assignment an isomorphism?

(b) Using (a), find $D_AF(X)$, for $X \in T_A \mathrm{Mat}(2;\mathbb{R})$.

(c) Now assume $F(A) = \Phi$. Why is D_AF not surjective? (Hint: what is $D_AF(X)^\top$?). Can you modify F so that D_AF becomes surjective?

(d) Again assuming $F(A) = \Phi$, what is $\ker D_IF$, and what is its dimension?

We call $F^{-1}(\Phi)$ the *split-orthogonal* group $O(1,1)$. It consists of transformations that preserve the quadratic form $x^2 - y^2$. In this exercise, we showed that $O(1,1)$ is a manifold, because it is the pre-image of a regular value (namely Φ) of $F : \mathrm{Mat}(2;\mathbb{R}) \to \mathrm{Sym}(2;\mathbb{R})$, and we showed $T_IO(1,1)$ is a one-dimensional subspace of $\mathrm{Mat}(2;\mathbb{R})$.

Exercise 11. View S^1 as the circle $x^2 + y^2 = 1$, inside $\mathbb{R}^2$. Define

$$\pi : \mathbb{R} \to S^1, \quad \pi(t) = (\cos(t), \sin(t)).$$

Let $X = f(t)\frac{\partial}{\partial t}$ be a vector field on $\mathbb{R}$.

(a) Express $T_{t_0}\pi(X_{t_0})$ in terms of $\frac{\partial}{\partial x}\Big|_{\pi(t_0)}$ and $\frac{\partial}{\partial y}\Big|_{\pi(t_0)}$.

(b) Express $T_{t_0}\pi(X_{t_0})$ as a linear map $C^\infty(S^1) \to \mathbb{R}$.

(c) Is there a vector field Y on S^1 that is π-related to:

(i) $\frac{\partial}{\partial t}$?

(ii) $e^t \frac{\partial}{\partial t}$?

(iii) $g(\pi(t))\frac{\partial}{\partial t}$, where $g : S^1 \to \mathbb{R}$?

Exercise 12. Suppose $F : M \to N$ is a diffeomorphism.

(a) Given a vector field X on M, what is the unique vector field Y on N that is F-related to X? Denote this vector field F_*X.

(b) Prove that

$$[F_*X_1, F_*X_2] = F_*[X_1, X_2]$$

where X_i are vector fields on M. Hint: the following commutes:

$$\begin{array}{ccc} C^\infty(M) & \xrightarrow{X_i} & C^\infty(M) \\ {\scriptstyle F^*}\uparrow & & {\scriptstyle F^*}\uparrow \\ C^\infty(N) & \xrightarrow{F_*X_i} & C^\infty(N). \end{array}$$

View S^2 as the unit sphere inside $\mathbb{R}^3$. We will construct a diffeomorphism of S^2 that fixes exactly one point.

Exercise 13. Let $N = (0,0,1)$ be the north pole. Recall that stereographic projection is a diffeomorphism $\varphi : S^2 \setminus \{N\} \to \mathbb{R}^2$ defined by

$$\varphi(x,y,z) := \left(\frac{x}{1-z}, \frac{y}{1-z}\right) = (u,v)$$
$$\varphi^{-1}(u,v) = \left(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, 1 - \frac{2}{u^2+v^2+1}\right) = (x,y,z).$$

(a) Let $\Phi : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ be the map $\Phi(t,(u,v)) := (u+t, v)$. This is a flow of a vector field U on $\mathbb{R}^2$. Find U.

(b) View φ^{-1} as a map $\mathbb{R}^2 \to \mathbb{R}^3$. It turns out there is a vector field X on $\mathbb{R}^3$ such that $U \sim_{\varphi^{-1}} X$. Find X.

(c) Prove that X is tangent to S^2. (Hint: $X_{(x,y,z)}$ should be perpendicular to what vector?)

(d) What are the integral curves of $X|_{S^2}$ in terms of Φ and φ^{-1}? Draw some integral curves. Is $X|_{S^2}$ complete?

(e) Where does $X|_{S^2}$ vanish? How can we use $X|_{S^2}$ to create a diffeomorphism of S^2 that only fixes one point?

We will discuss some aspects of problem 5 from Homework 4 in some more detail. Let

$$E = \sum_{i=1}^{m} x^i \frac{\partial}{\partial x^i}$$

be the Euler vector field on $\mathbb{R}^m$. It is complete, with flow $\Phi_t(x) = e^t x$.

Exercise 14. First, we take $m = 1$.

(a) For $f \in C^\infty((0,\infty))$, write the equation $L_E f(x) = kf(x)$ in terms of f and f'.

(b) You should have an ODE. Solve it (Hint: the solution will depend on $f(1)$). Is f necessarily a polynomial on $(0,\infty)$?

(c) What if $f \in C^\infty(\mathbb{R} \setminus \{0\})$? Or if $f \in C^\infty(\mathbb{R})$?

Exercise 15. Now we deal with general m.

(a) For $f \in C^\infty(\mathbb{R}^m \setminus \{0\})$, write the equation $L_E f(x) = kf(x)$ in terms of f and ∇f.

(b) You should have a PDE, which is harder to solve. Lets try to uncover an ODE similar to Exercise 1 instead.

If we replace x with tx_0 in the PDE in (a), can you write the resulting equation as an ODE in t? What is its solution? What does this say about f?

(c) Now assume $f \in C^\infty(\mathbb{R}^m)$. Why is k a non-negative integer?

(d) Prove that $\frac{\partial f}{\partial x_i}$ satisfies $L_E \frac{\partial f}{\partial x_i} = (k-1)\frac{\partial f}{\partial x_i}$. Why may we conclude f is a polynomial?

Fix a manifold M of dimension m. Given a 1-form $\alpha \in \Omega^1(M)$, we can pull it back by any function $F : \mathcal{U} \to M$ (here $\mathcal{U}$ and $\mathcal{V}$ always denote a subset of $\mathbb{R}^m$) to get a 1-form $F^*\alpha \in \Omega^1(\mathcal{U})$. Think of this as an assignment:

To each smooth function $F : \mathcal{U} \to M$, assign the 1-form $F^*\alpha \in \Omega^1(\mathcal{U})$.

In this tutorial, we investigate going the other way. Let a be an assignment as follows:

To each smooth function $F : \mathcal{U} \to M$, assign a 1-form $a(F) \in \Omega^1(\mathcal{U})$.

We will say that a is *induced* by α if $a(F) = F^*\alpha$ for all $F : \mathcal{U} \to M$.

Exercise 16.

(a) Not every assignment a is induced by some 1-form α. Give an example.

(b) If we try to find some α that induces a, we might try to define α in each coordinate chart. For $\varphi : U \to \varphi(U)$, how should we try to define $\alpha|_U$? How would we define α_p?

(c) In order for α to be well-defined, it should not depend on the choice of coordinate charts. What condition on a would make this so?

(d) Given a satisfying the condition in (c), prove that α constructed in (b) is a 1-form that induces a (you may assume, and will need, Lemma 7.11). Propose an alternate definition of a 1-form.

In this tutorial, we will warm up with a proof of the Poincaré lemma in a special case. Then we will prove the Poincaré lemma in generality. First, the lemma states:

Lemma (Poincaré)**.** *For all $n \geq 0$, and $k \geq 1$,*

$$H^k(\mathbb{R}^n) = 0.$$

In other words, every k-form is exact (i.e. for every $\alpha \in \Omega^k(\mathbb{R}^n)$, there exists $\beta \in \Omega^{k-1}(\mathbb{R}^n)$ such that $d\beta = \alpha$).

Exercise 17. We first prove this lemma for $n = 2$ and $k = 1$. Let $\alpha = Pdx + Qdy$ be a closed 1-form on $\mathbb{R}^2$. We seek a function β so that $d\beta = \alpha$.

(a) Write the equations $d\beta = \alpha$ and $d\alpha = 0$ in terms of their component functions.

(b) If $f : \mathbb{R} \to \mathbb{R}$ is a function, an anti-derivative is $F(t) := \int_0^t f(s)ds$. What is a function $\mathbb{R}^2 \to \mathbb{R}$ whose x-partial equals $P(x, y)$?

(c) The y-partial of your answer to (b) is probably not $Q(x, y)$, but is close. Modify the function in (b) to get a primitive for α (Hint: you do not want to change the x-partial).

In this tutorial, we will use Stokes' theorem to find a volume (area) form on the sphere. View the unit sphere S^2 as inside $\mathbb{R}^3$.

Exercise 18.

(a) The differential form $dx \wedge dy \wedge dz \in \Omega^3(\mathbb{R}^3)$ is closed. In fact, it is exact (why?). Find some primitives ω.

(b) Find a primitive ω which never vanishes on S^2. We call a top degree differential form that never vanishes a *volume form.*

(c) Evaluate $\int_{S^2} \omega$ (where S^2 is oriented with outward-pointing normals). Is this the surface area of the sphere? What if S^2 has radius $r \neq 1$?

(d) Prove *Archimedes' theorem*: if D is the region of the sphere between the hyperplanes $z = a$ and $z = b$, for $-1 < a < b < 1$ then the surface area of D depends only on $b - a$ (Hint: cylindrical coordinates).