**Exercise 1.** Please read Spivak's Chapter 1, "Basic properties of numbers," before this exercise. Determine if the following statement is true or false, and explain your answer in one sentence.

- Spivak's proof that $a - b = b - a$ implies $a = b$ relies only on the arithmetic axioms, which are: associativity, commutativity of addition and multiplication, neutral elements for addition and multiplication, inverse elements for addition and multiplication, and the distributive law.

**Exercise 2.** Read Spivak Chapter 25, "Complex Numbers." Identify the pages in the chapter in which he discusses cubic equations. You may then skip these pages. Find all complex numbers $z$ such that $z^2 = i$. Write your answer(s) $z$ in the form $z = a + ib$.

**Exercise 3.** Read Spivak Chapter 2, "Numbers of Various Sorts." Select all the statements Spivak **proves** in this chapter:

(a) There is a real number $x$ such that $x^2 = 2$, and this $x$ is not rational.

(b) If a real number $x$ is such that $x^2 = 2$, then $x$ is not rational.

(c) There is a rational number $x$ such that $x^2 = 2$.

(d) Every number is even or odd.

In one sentence, justify your selection.

**Exercise 4.** Read Spivak Chapter 8, "Least Upper Bounds," but identify and skip the part where he proves theorems from Chapter 7. Determine if each statement is true or false, and in one sentence explain why.

(a) $\mathbb{Q}$ has the least upper bound property.

(b) $\mathbb{N}$ does not have a least upper bound.

(c) The empty set $\emptyset$ has an upper bound.

(d) The empty set $\emptyset$ has a supremum.

**Exercise 5.** This is also based on Chapter 8. Give (without proof) the infimum and supremum of the following sets, if they exist:

$$A := (-\sqrt{5}, 2], \quad B := [-\sqrt{5}, \infty), \quad C := \{-2\} \cup (-\sqrt{5}, \sqrt{7}).$$

**Exercise 6.** Read Spivak Chapter 3, "Functions."

(a) Suppose $f$ and $g$ are functions, and $A :=$ domain $f$, and $B :=$ domain $g$. Write the domain of: $f + g$, $f \cdot g$, $f/g$, $cf$ (for $c \in \mathbb{R}$, and $f \circ g$.

(b) Define $f$ by $f(x) := \sqrt{x}$, and $g$ by $g(x) := -1 - x^2$. What is the domain of $f$? of $g$? of $f \circ g$? of $g \circ f$?

**Exercise 7.** (a) How does Spivak define the distance between two points $(a, b)$ and $(c, d)$ in the plane? Do we need the least upper bound property of real numbers for Spivak's definition to make sense?

(b) Draw the graph of some function $f$ where:

- $f(x) \leq 4$ for all $x \in$ domain $f$,
- $f(x) \neq 0$ for all $x \in$ domain $f$,
- if $x < -2$, then $x$ is not in the domain of $f$.

**Exercise 8.** Read Spivak Chapter 5, "Limits."

(a) Let $f$ be a function, and $x_0, L \in \mathbb{R}$. Write the formal definition, and also the notation, for, "$f$ approaches the limit $L$ near $x_0$." (Remember the definition should read like a complete sentence, without any unquantified / ambiguous terms).

(b) Is the following true or false? If true, explain briefly why. If false, slightly modify the statement so that it is true.

If $f$ approaches the limit $L$ near $x_0$, then there exists an open interval $I$ such that $x_0 \in I$ and $I \subseteq$ domain $f$.

**Exercise 9.** Read Spivak Chapter 6, "Continuity." Consider the function $f$ defined on the interval $(0, 1)$ by

$$f(x) := \begin{cases} 0 & \text{if } x \text{ is irrational} \\ \frac{1}{q} & \text{if } x = \frac{p}{q} \text{ in lowest terms.} \end{cases}$$

(a) At what points is this function continuous? You do not need to justify your answer.

(b) By definition $f(3/4) = 1/4 > 0$. Is there an open interval $I$ around (i.e. containing) $1/4$ such that for every $x \in I$, $f(x) > 0$? Briefly state why or why not.

(c) Explain why your answer to (b) does not contradict Theorem 3 from Chapter 6.

**Exercise 10.** Read Spivak Chapter 4, Appendix 1, "Vectors." Consider the vectors $e := (0, 1)$ and $f := (1, 1)$.

(a) Given a vector $v = (v_1, v_2)$, can you find $a, b \in \mathbb{R}$ such that $v = ae + bf$?

(b) Suppose $a, b \in \mathbb{R}$ and $ae + bf = (0, 0)$. Is it possible that both $a \neq 0$ and $b \neq 0$?

If you take a course in linear algebra, you will recognize the set $\{e, f\}$ as a *basis* of the plane, when viewed as a *vector space*.

**Exercise 11.** Read Spivak Chapter 7, "Three Hard Theorems." Fill in the proof of the following theorem.

**Theorem.** Suppose $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$, where $n > 0$ is even and $a_0 = f(0) < 0$. Then there are at least two numbers, $x_1$ and $x_2$, such that $f(x_1) = 0$ and $f(x_2) = 0$. In other words, $f$ has at least two zeroes.

*Proof.* Choose

$$M := \max(\underline{\qquad}).$$

Then if $|x| > M$, we have $|x^k| > |x|$ and also

$$\frac{|a_{n-k}|}{|x^k|} < \frac{|a_{n-k}|}{|x|} < \underline{\qquad} = \frac{1}{2n}.$$

Consequently, by the _____ and the inequality above,

$$\left|\frac{a_{n-1}}{x} + \frac{a_{n-2}}{x^2} + \cdots + \frac{a_0}{x^n}\right| \leq \underline{\qquad} < \frac{1}{2n} + \cdots \frac{1}{2n} = \frac{1}{2}.$$

This implies that, for $|x| > M$,

$$0 < \frac{x^n}{2} \leq x^n\left(1 + \frac{a_{n-1}}{x} + \frac{a_{n-2}}{x^2} + \cdots + \frac{a_0}{x^n}\right) = f(x).$$

In particular, if we choose $a :=$ _____ and $b :=$ _____, then $a > 0$ and $f(a) > 0$, and $b < 0$ and $f(b) > 0$. Because $f(0) < 0$, applying Theorem _____ to the interval _____ lets us conclude there is some $x_1 > 0$ such that $f(x) = 0$. Similarly, applying Theorem _____ to the interval _____ lets us conclude there is some $x_2 < 0$ such that $f(x) = 0$. As $x_2 < 0 < x_1$, we have found two distinct zeroes of $f$, as desired. ☐

**Exercise 12.** Read Spivak Chapter 8, "Least Upper Bounds."

(a) In the proof of Theorem 7-1 (the Intermediate Value Theorem), Spivak uses:

**Lemma.** Let $A \subseteq \mathbb{R}$ be a non-empty set, bounded from above by $\alpha$. Then $\alpha = \sup A$ if and only if for every $\epsilon > 0$, there is some $x \in A$ such that $\alpha - \epsilon < x \leq \alpha$.

Prove this lemma, and indicate in which paragraph Spivak uses it.

(b) Suppose $f$ is continuous from the right at $a$. Prove there is a $\delta > 0$ such that $f$ is bounded on the set $[a, a+\delta)$. Must $f$ be bounded on some set of the form $(a - \delta', a + \delta')$?

**Exercise 13.** Read Spivak Chapter 9, "Derivatives."

(a) Let $f(x) := x^2 \sin \frac{1}{x}$, where $f(0) := 0$. Consider the following argument that $f'(0)$ does not exist:

*Proof.*

$$\begin{aligned} f'(0) &= \lim_{h\to 0} \frac{f(h)-f(0)}{h} && \text{by definition of } f'(0) \\ &= \lim_{h\to 0} \frac{h^2 \sin\frac{1}{h}}{h} && \text{by definition of } f \\ &= \left(\lim_{h\to 0} h\right)\left(\lim_{h\to 0} \sin\frac{1}{h}\right) && \text{by the product rule for limits,} \end{aligned}$$

which does not exist because $\lim_{h\to 0} \sin\frac{1}{h}$ does not exist. □

Is this argument correct? If not, point out the error, and say whether $f'(0)$ exists or not.

(b) For each item below, give a function $f : [-1, 1] \to \mathbb{R}$ (with an expression or a clear drawing) with that property:

- $f$ is not differentiable at any point (note we do not require continuity, so keep it simple).
- $f$ is continuous, and both the left and right-hand derivative of $f$ at 0 diverges to $\infty$.
- The left-hand derivative of $f$ does not exist at 0 (and does not diverge to $\pm\infty$), and the derivative of $f$ at $\frac{1}{2}$ is 1.

**Exercise 14** (Complete Exercises 1 and 2 first)**.** Read Spivak Chapter 8, Appendix, "Uniform Continuity."

(a) Define what it means for $f$ to be uniformly continuous on an interval $A$.

(b) In class, we defined what it means for $f$ to be *Lipschitz* on $A$. Give the definition here.

(c) Suppose $f$ is a function on $[a, b]$. Connect the concepts below with "implies" or "does not imply" arrows.

continuous uniformly continuous

Lipschitz

**Exercise 15.** Read Spivak Chapter 10, "Derivatives."

(a) Using the product, quotient, sum, and chain rules, compute the derivative of:

- $f(x) := \sin(\cos(x^2))$.
- $f(x) := \frac{1+x^2 \cos x}{x - \sin x}$.

(b) In what assignment did we prove: the function $h(x) := \sqrt{x}$ is differentiable at all $x > 0$, and $h'(x) = \frac{1}{2\sqrt{x}}$? Using this fact, differentiate $f(x) := \sqrt{\frac{x+1}{x-1}}$ (be careful when simplifying).

(c) Define

$$F(x) := \begin{cases} \frac{1}{x} + 2 & \text{if } x > 0 \\ \frac{1}{x} + 7 & \text{if } x < 0. \end{cases}$$

What is $F'(x)$? Can you find another function $G$ such that $G'(x) = F'(x)$? [Extra: Can you guess what every such $G$ might be defined?]

**Exercise 16.** Read Spivak Chapter 12, "Inverse Functions," up to but not including Theorem 4. Consider the function $f(x) := \sqrt{|x|}$. Draw it.

(a) Does $f$ have an inverse? If so, give it and draw it. If not, find an interval $A \subseteq \mathbb{R}$ on which $f|_A$ does have an inverse, give it and draw it. Can we conclude that $(f|_A)^{-1}$ is continuous without appealing to the formula for $(f|_A)^{-1}$ (i.e. are there any helpful theorems we can use)?

(b) Is there any point $x$ in $\mathbb{R}$ such that for every $\delta > 0$, the restriction $f|_{(x-\delta, x+\delta)}$ has no inverse?

**Exercise 17.** Read Spivak Chapter 11, "Significance of the Derivative." Exercises 2 and 3 are also on this chapter.

(a) Suppose $f$ is a function on $[a, b]$. What are three kinds of points where its maximum/minimum can occur? (i) (ii) (iii) . Can its max/min occur anywhere else?

(b) Draw a function whose max occurs at: only points in (i); only points in (ii); only points in (iii); at points in (i), (ii), and (iii). So you have to draw 4 graphs in total.

**Exercise 18.** Is each statement true or false? Justify in one sentence.

(a) If $f$ is strictly increasing on $(a, b)$, then $f'(x) > 0$ for all $x \in (a, b)$.

(b) Using Rolle's theorem, Spivak proves by induction that every $n$th degree polynomial has exactly $n$ roots.

(c) If $f''(a) > 0$, then $f$ has a local minimum at $a$.

**Exercise 19.** For each function, state whether we can use Theorem 7[(1)] to conclude that $f'(a)$ exists. If so, use Theorem 7 to compute $f'(a)$. If not, explain why.

(a)

$$f(x) = \begin{cases} (x-1)^2 & \text{if } x \geq 1 \\ -(x-1)^2 & \text{if } x < 1, \end{cases} \quad \text{at } a = 1$$

[(1)] This theorem probably seems narrow in its application, but in fact it recently came up in my own research. I was studying a 1986 result called "Joris' Theorem," and came across a proposed proof from 2003 which used only tools we will see in this course. One of them was Theorem 7, but I did not recognize it from Spivak and spent time struggling with it on my own. So, reading Spivak carefully truly is worth it! Incidentally, it turned out that 2003 proof was flawed, and other techniques are needed to prove Joris' theorem.

(b)

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0, \end{cases} \quad \text{at } a = 0$$

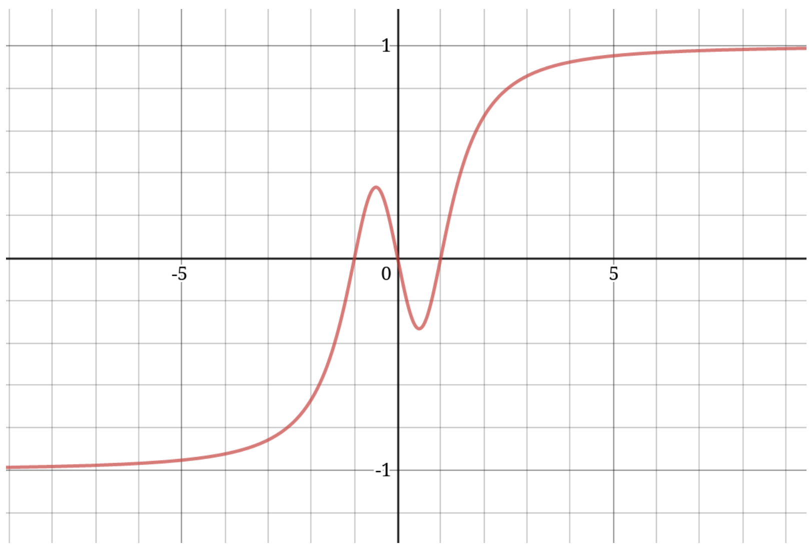
(Hint: does $\lim_{x \to 0} f'(x)$ exist?)

(c)

$$f(x) = \begin{cases} \frac{\sqrt{x+1}-1}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0, \end{cases} \quad \text{at } a = 0$$

(Hint: is $f$ continuous at 0?)

**Exercise 20.** Read Spivak Chapter 11, Appendix, "Convexity and Concavity." Consider the following graph of a function.




Assume this is the graph of $f'$ (the *derivative* of $f$).

(a) On what interval(s) is $f$ convex? Concave? Where are its inflection point(s)?

(b) Does $f$ have local extrema? If so, where?

(c) Is $f'$ odd, even, or neither? Can you guess whether $f$ is odd, even, or neither? c.f. Spivak problems 9.23 and 9.24.

[Extra: graph $f$.] [Bonus extra: can you find the function I used to get the above graph? Desmos or geogebra are useful, free, online graphing tools that might help. Ask me for hints!]

**Exercise 21.** Read Spivak Chapter 12, "Inverse Functions." Consider the function defined on $(0, 2)$ by

$$f(x) := \begin{cases} x^2 & \text{if } x \text{ is rational} \\ 2x - 1 & \text{if } x \text{ is not rational.} \end{cases}$$

(a) Does Spivak's version of the inverse function theorem, Theorem 5, apply to $f$ at the point $x = 1$? Why or why not?

(b) Is $f^{-1}$ differentiable at 0? If so, what is $(f^{-1})'(0)$? Explain briefly, without a proof.

**Exercise 22.** Read Spivak Chapter 12, Appendix, "Parametric Representation of Curves." Consider two parametrizations, for $t \in [-\pi, \pi)$

$$c_1(t) := (\cos t, 2\sin t)$$
$$c_2(t) := (\cos t^3, 2\sin t^3).$$

(a) Sketch the curve(s) parametrized by $c_1$, and by $c_2$. (Hint: begin by thinking about the similar parametrization $(\cos t, \sin t)$).

(b) According to Spivak's definition, does $c_1$ have a tangent line through $c_1(0)$? Does $c_2$ have a tangent line through $c_2(0)$? For each of these, give the equation if the tangent exists.

**Exercise 23.** Read Spivak Chapter 13, "Integrals."

(a) When he proves $\int_0^b x dx = \frac{b^2}{2}$, does Spivak evaluate $L(f, P)$ and $U(f, P)$ for *any* partition $P$, or only some special partitions?

(b) Let $B := \{-2, 0, 1, 3\}$. Is the function $f(x) := (1 - 1_B(x))x^2 \sin \frac{1}{x}$ (taking $f(0) := 2$) integrable on $[-4, 5]$? Do not make any computations!

(c) Suppose $f$ is integrable on $[a, b]$. Define the functions on $[a, b]$:

$$F(x) := \int_a^x f(y)dy, \quad G(z) := \int_a^z f(x)dx, \quad H(t) := \int_a^t f(V)dV.$$

Are these all the same function?

**Exercise 24.** Read Spivak Chapter 13, Appendix, "Riemann Sums." Set $f(x) := e^x$ on $[-1, 2]$. Let $P := \{-1, -0.5, 0.5, 1.5, 2\}$, a partition of $[-1, 2]$. Write two Riemann sums of $f$ for $P$. You do not need to simplify.

[Extra: what are the sums $L(f, P)$ and $U(f, P)$? Evaluate everything with a calculator and "verify" that $L(f, P) \leq$ the Riemann sums $\leq U(f, P)$].

**Exercise 25.** Read Spivak Chapter 14, "The Fundamental Theorem of Calculus."

(a) Consider the statement

Every ____________ function $f$ on $[a, b]$ is the derivative of some function.

For the statement to be true, can we fill the blank with

(i) Continuous?

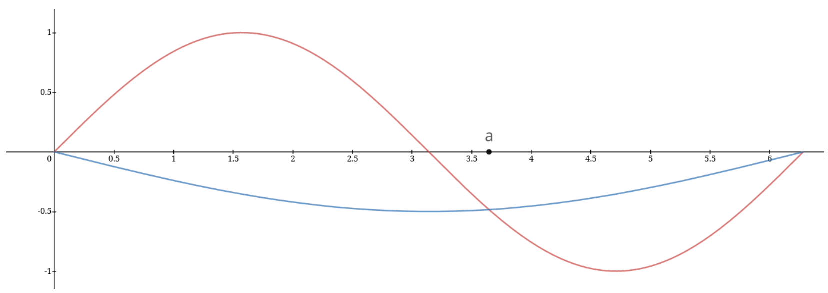
(ii) Lipschitz?

(iii) Integrable?

(iv) Bounded?

When the statement is true, give some function $F$ such that $F' = f$. [Extra: can you find $F$ satisfying $F\left(\frac{b-a}{2}\right) = 0$?]. When it is false, give a counter-example. Do not justify.

(b) On $[0, 2\pi]$, define $f(x) := \sin x$ and $g(x) := -\frac{1}{2}\sin\left(\frac{1}{2}x\right)$. Shade the region bounded by the graphs of $f$ and $g$. Which integral(s) give its area?




$$I_1 := \int_0^{2\pi} f - g, \quad I_2 := \int_0^{2\pi} |f - g|, \quad I_3 := \int_0^a f - g + \int_a^{2\pi} g - f, \quad I_4 := -\int_{2\pi}^0 |f - g|$$

Which one would you choose to actually compute the area explicitly?

**Exercise 26.** Read Spivak Chapter 18, "The Logaritm and Exponential Functions."

(a) For rational $x$, does Spivak first *define* $e^x$ as $\exp(x)$, or does he first *deduce* $e^x = \exp(x)$ from the defintion of $e$ and properties of exp?

(b) True of false: exp is the only function such that $f(x+y) = f(x)f(y)$ and $f(1) = e$. If true, briefly justify. If false, what extra condition on $f$ can we impose to guarantee $f = \exp$?

(c) A function $f : I \to \mathbb{R}$ is called *smooth* if it is $C^k$ for every $k$. For example, polynomials are smooth. Is

$$f(x) := \begin{cases} e^{-\frac{1}{x^2}} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

smooth? [Extra: prove there is no polynomial $p(x) = a_n x^n + \cdots + a_0$ such that $p = f$].

**Exercise 27.** Read Spivak Chapter 15, "The Trigonometric Functions," excluding the middle of page 305 to the middle of page 308 (where Spivak defines sin; we take a different approach).

(a) Fill in the blank:

For each $x$ in _____, $\alpha := \arccos x$ is the unique value in _____ such that $\cos(\alpha) = x$.

(b) Is $\arccos(\cos(7\pi/6)) = 7\pi/6$? If so, say why and if not, give the correct value.

(c) Use the chain rule to find $(\arccos \circ \cos)'(\alpha)$ for $\alpha \notin \{k\pi \mid k \in \mathbb{Z}\}$ (why the restriction on $\alpha$?). [Extra: can you graph $\arccos \circ \cos$?]

**Exercise 28.** Read Spivak Chapter 19, "Integration in Elementary Terms." Consider the definite integral $\int_{-1}^{1} \sqrt{1+x^2}dx$.

(a) Consider this attempt to evaluate the integral. Is it correct? If not, where is the error?

Set $u = 1+x^2$. As $du = 2xdx$ does not appear in the integrand, we instead find $dx$ by solving for $x$:

$$\begin{aligned} u &= 1 + x^2, \quad \text{on } [-1, 1] \\ x &= \sqrt{u-1} \\ dx &= \frac{1}{2\sqrt{u-1}}du. \end{aligned}$$

So we know what to replace $1 + x^2$ and $dx$ with. Finally, since the $x$ bounds are from $-1$ to 1, the $u$ bounds must be from $1 + (-1)^2 = 2$ to $1 + 1^2 = 2$. All together, we get

$$\int_{-1}^{1} \sqrt{1+x^2}dx = \int_{2}^{2} \frac{\sqrt{u}}{2\sqrt{u-1}}du = 0,$$

since the RHS has both endpoints the same.

[Hint: draw the area calculated by the definite integral. Also, a parenthetical comment by Spivak is relevant.]

(b) Why is it true that

$$\int_{\frac{1}{2}}^{1} \sqrt{1+x^2} = \int_{\frac{5}{4}}^{2} \frac{\sqrt{u}}{2\sqrt{u-1}}du?$$

**Exercise 29.** Read Spivak Chapter 19, Appendix, "The Cosmopolitan Integral." Consider the functions

$$f(x) := \frac{1}{x}, \quad g(x) := \frac{1}{4}, \quad \text{on } [1, 2].$$

Let $V_f$ and $V_g$ be the solids of revolution obtained by rotating $f$ and $g$ around the $x$-axis, respectively.

(a) What familiar shape is $V_g$? Is $V_f$ contained in $V_g$, or vice versa?

(b) Suppose we take $V_f$ and remove the portion $V_g$ (imagine drilling a hole in $V_f$). Call this new shape $V$. Which integral equals the volume of $V$?

$$I_1 := \pi \int_1^2 \left(\frac{1}{x} - \frac{1}{4}\right)^2 dx, \qquad I_2 := \pi \int_1^2 \left(\frac{1}{x^2} - \frac{1}{16}\right) dx, \qquad I_3 := 2\pi \int_1^2 \left(1 - \frac{x}{4}\right) dx.$$

[Extra: one of the other integrals corresponds to the volume of a different solid obtained from $f$ and $g$. Describe this solid.]

**Exercise 30.** Read Spivak Chapter 22, "Infinite Sequences." Consider the sequence $a_n := e^{\cos(n\pi/6)}$.

(a) Is $A := \{a_n \mid n \in \mathbb{N}\}$ bounded. How many elements does it have?

(b) Does $\{a_n\}$ contain a strictly increasing or decreasing subsequence? Give one or justify why none exists. [Hint: part (a) should help].

(c) List the peak points of $\{a_n\}$, if they exist.

(d) Does $\{a_n\}$ contain a weakly increasing or weakly decreasing subsequence? Give one (and its limit) or justify why none exists.

(e) [Extra: modify $\{a_n\}$ so it converges.]

**Exercise 31.** Read Spivak Chapter 23, "Infinite Series."

(a) Each argument below is wrong. In each case, point out the convergence test attempted, and where the error lies.

(i) Set $a_n = 1/n^2$ and $b_n = 1/n$. Since $0 \leq a_n \leq b_n$ for all $n$, and $\sum_{n=1}^{\infty} a_n$ converges, so does $\sum_{n=1}^{\infty} b_n$.

(ii) Set $a_n = \frac{1}{n!e^n}$. Since $a_n > 0$ for all $n$, and

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{1}{e(n+1)} = 0,$$

the convergence test we are using is inconclusive.

(iii) Set $a_n = \frac{\log n}{n}$, for $n \geq 4$. Define $f(x) := \frac{\log x}{x}$, on $[4, \infty)$. Since $f$ is positive and decreasing on $[4, \infty)$, and $\int_4^{\infty} f(x)dx$ converges, so does $\sum_{n=4}^{\infty} a_n$.

(iv) Set $a_n = \frac{1}{\sqrt{n}}$ and $b_n = \frac{1}{n^2}$. Since

$$\lim_{n\to\infty} a_n/b_n = \lim_{n\to\infty} n^{3/2} = \infty \neq 0,$$

and since $\sum_{n=1}^{\infty} b_n$ converges, so does $\sum_{n=1}^{\infty} a_n$.

(b) Set $a_n = \frac{1}{\sqrt{n}}$. Justify whether the following is true or false: The infinite series

$$1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \cdots = \sum_{n=1}^{\infty} (-1)^{n+1} a_n$$

converges, say to $S$. Moreover, every rearrangement of $\{(-1)^n a_n\}$ converges to the same number $S$.

**Exercise 32.** Read Spivak Chapter 24, "Uniform Convergence and Power Series."

(a) Suppose $(f_n)$ is a sequence of functions on $[0, 1]$ such that $\lim_{n\to\infty} f_n(x)$ exists for each $x$; denote each limit by $f(x)$. Justify whether each statement below is true or false.

(i) If there is a sequence of points $(x_n)$ such that $\lim_{n\to\infty} f_n(x_n)$ diverges to infinity, then there must be some $x$ such that $f(x) > 0$ (i.e. the limit cannot be the zero function).

(ii) If each $f_n$ is $C^1$, and $(f_n)$ converges to $f$, and $(f_n')$ converges uniformly to $g$, then $f$ is differentiable and $f'(x) = g(x)$.

(iii) If each $f_n$ is continuous, and $(f_n)$ converges to $f$ uniformly, then $f$ is uniformly continuous.

(b) Give a sequence $(f_n)$ of non-constant differentiable functions on $\mathbb{R}$ that satisfy the hypotheses of the Weierstrass $M$-Test. (Hint: you could try to find a smooth analog to Spivak's example). Explain why your choice works.

(c) Fix sequences $(a_n)$ and $(b_n)$.

(i) Suppose $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} b_n = l$, for some $l$. Why do $\sum_{n=1}^{\infty} a_n x^n$ and $\sum_{n=1}^{\infty} b_n x^n$ converge on every closed subinterval of $(-1, 1)$? Must they necessarily converge on $[-1, 1]$?

(ii) Further assume that $\sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty} b_n x^n =: f(x)$. Prove that $a_k = b_k$ for each $k$.

(d) [Extra: Consider the functions $f_n(x) = \frac{1}{2^n}\{2^n x\}$ on $[0, 1]$, where $\{x\}$ denotes the distance from $x$ to the nearest integer, as in Spivak. The functions converge uniformly to the zero function, whose graph has arclength 1. What is the arclength of each graph of $f_n$ (say parametrized by $\gamma(t) := (t, f_n(t))$)? What does this mean for the relation between uniform convergence and arclength?]

**Exercise 33.** Read Spivak Chapter 20, "Approximation by Polynomial Functions." Suppose $f$ is at least twice differentiable on $\mathbb{R}$, and

$$\begin{aligned} f'' - f &= 0 \\ f(0) &= 0 \\ f'(0) &= 0. \end{aligned}$$

We will prove $f = 0$, in a similar way to Spivak's arguments about solving $f'' + f = 0$ near the end of the chapter (find them!).

*Proof.* The set $\{f^{(k)} \mid k \in \mathbb{N}\}$ has _____ elements [Hint: $f^{(3)} = (f'')' = f'$, by assumption on $f$. What about $f^{(4)}$?] Moreover,

$$f^{(k)} = \begin{cases} \_\_\_\_\_ & \text{if } k \text{ is } \_\_\_\_\_ \\ \_\_\_\_\_ & \text{if } k \text{ is } \_\_\_\_\_. \end{cases}$$

In particular, $f^{(k)}(0) = \_\_\_\_\_$ for all $k$. Fix $n$. This means $P_{n,0}(x) = \_\_\_\_\_$. On the other hand, by Taylor's theorem, assuming for the moment $x > 0$,

$$R_{n,0}(x) = \_\_\_\_\_, \text{ for some } t \in [0, x].$$

[Spivak writes unclearly here; he has $a$ when it should be 0.] Since $f = P_{n,0} + R_{n,0}$, we conclude $f = \_\_\_\_\_$.

Now, $f$ and $f'$ are continuous, so by the boundedness theorem on $[0, x]$, there exists some $M_0$ and $M_1$ such that

$$|f(t)| \leq M_0, \quad |f'(t)| \leq M_1, \quad \text{for all } t \in [0, x].$$

Therefore, $|f^{(n+1)}(t)| \leq \_\_\_\_\_$ for all $t \in [0, x]$ [Hint: it should probably be bigger than both $M_0$ and $M_1$.] In particular, we can bound

$$R_{n,0}(x) \leq \_\_\_\_\_.$$

For any $\varepsilon > 0$, we may therefore choose $n$ so that $R_{n,0}(x) = \_\_\_\_\_ < \varepsilon$. In other words, $|f(x)| < \varepsilon$ for all $\varepsilon > 0$, which means $|f(x)| = 0$. Since $x$ was arbitrary and positive, we conclude $f = 0$ on $[0, \infty)$. A similar argument holds on $(-\infty, 0]$.

[Aside: this implies that given $a, b$, there is a unique function $f$ satisfying $f'' - f = 0$ and $f(0) = a$ and $f'(0) = b$. Choosing $a = 0$ and $b = 1$, the unique solution $f$ is called the "hyperbolic sine function," denoted sinh. Choosing $a = 1$ and $b = 0$, the unique solution $f$ is called the "hyperbolic cosine function," denoted cosh.] $\square$